Upsampling and Downsampling Contd.

Consider the operation of upsampling a discrete–time signal $x[n]$ by a factor of L followed by a downsampling operation by a factor of M. Denote the output of this system $y_1[n]$. In a similar fashion consider the system that comprises of first a downsampling operation of M followed by an upsampling operation by L. Denote the output of this system by $y_2[n]$.

Our goal is to investigate the conditions on M and L so that the upsampling operation and downsampling operation commute. First note that the downsampling operation by a factor M is the right inverse of the upsampling operation by a factor of M, i.e.,

$$\downarrow M\ (\uparrow M\ (x[n])) = \mathbf{I}(x[n]).$$

The same relation in the reverse order is not true, i.e.,

$$\uparrow M\ (\downarrow M\ (x[n])) \neq \mathbf{I}(x[n]).$$

This implies that the upsampling operation by a factor M does not commute with the downsampling operation by the factor M.

Relating the output of the first system $y_1[n]$ interms of the input $x[n]$ we have:

$$\boxed{y_1[n] = \sum_{p=-\infty}^{\infty} x[p]\delta[Mn - pL].} \tag{1}$$

In a similar fashion relating the output of the second system $y_2[n]$ we have:

$$\boxed{y_2[n] = \sum_{p=-\infty}^{\infty} x[Mp]\delta[n - pL].} \tag{2}$$

In the Z-domain, we can relate $Y_1(z)$ to $X(z)$ via:

$$\boxed{Y_1(z) = \frac{1}{M}\sum_{p=0}^{M-1} X(z^{\frac{L}{M}} W_M^{pL}).} \tag{3}$$

$$\boxed{Y_2(z) = \frac{1}{M}\sum_{p=0}^{M-1} X(z^{\frac{L}{M}} W_M^{p}).} \tag{4}$$

If the two systems are equivalent then the set of complex "twiddle factors", $\{W_M^{pL},\ \ 0 \leq p \leq M-1\}$ should be equivalent to the set of "twiddle factors", $\{W_M^{q},\ \ 0 \leq q \leq M-1\}$ This set equivalence is possible only when the following equation

$$\boxed{\frac{2\pi pL}{M} = \frac{2\pi q}{M} - 2r\pi,\ 0 \leq p \leq M-1,\ 0 \leq q \leq M-1, r \in I.} \tag{5}$$

has a unique solution for the index pair (p, r) for every q in the integer range $0 \leq q \leq M-1$. In other words we require that the equation

$$\boxed{pL + Mr = q,\ 0 \leq p \leq M-1,\ 0 \leq q \leq M-1,\ r \in I.} \tag{6}$$

have a unique solution for a (p, r) pair for every q. This condition is possible only when the integers M and L are mutually prime or coprime, i.e.,

$$\gcd(M, L) = 1.$$

In other words for the upsampling operation by a factor of L to commute with the downsampling operation by a factor M, the integers M and L have to be mutually prime.

We can now see in this framework that the earlier result of the upsampling operation by the factor M not commuting with the downsampling operation by the factor M is consistent because $\gcd(M, M) = M$.