Non Uniform Quantization

In our previous discussion, we saw that when the input source signal $x[n]$ is uniformly distributed all the quantization intervals were of the same width, i.e, the source does not prefer any particular quantization interval. This may not be true in general for a source with an arbitrary distribution of values. In this general case it would make more sense to assign more levels in the ranges of values that occur more often and fewer quantization levels to ranges that are infrequent. This type of quantization is referred to as *non uniform quantization*.

For the purposes of this discussion let us assume that the source signal $x[n]$ is a *wide sense stationary* (WSS) random signal that has a first–order *probability density function* (PDF), $f_X(x)$. The mean μ_x and the variance σ_x^2 of the signal $x[n]$ are obtained via:

$$\mu_x = \int_{-\infty}^{\infty} x f_X(x)dx, \quad \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x)dx.$$

We would like to define a quantizer with Q levels of the following form:

$$x_q[n] \quad = \quad \begin{cases} m_i & x[n] \in [x_i, x_{i+1}] \\ -\infty & x[n] > x_Q \\ \infty & x[n] < x_2 \end{cases} \tag{1}$$

The design of the quantizer therefore boils down to the problem of determining the sequence of quantizer interval boundaries $\{x_i,\ i = 2, 3, \ldots, Q\}$. and the sequence of quantization output levels $\{m_i,\ i = 1, 2, \ldots, Q\}$.

The performance metric that we intend to minimize to determine the sequence of quantization levels and interval boundaries is the quantization noise power given by:

$$\sigma_e^2 = E\left\{(x_q[n] - x[n])^2\right\} = \int_{-\infty}^{\infty} (x_q - x)^2 f_X(x)dx.$$

Splitting this into the sum of integrals over the different quantizaion intervals the performance metric can be rewritten as:

$$\boxed{\sigma_e^2 = \sum_{i=1}^{Q} \int_{x_i}^{x_{i+1}} (m_i - x)^2 f_X(x)dx.} \tag{2}$$

We seek the set of parameters that minimize this performance metric. Taking the partial derivative with respect to the quantizer output levels, i.e.,

$\{m_i,\ i = 1, 2, \dots, Q\}$ and interchanging the expectation and derivative operations we obtain:

$$\boxed{\frac{\partial \sigma_e^2}{\partial m_i} = 0 \iff \int_{x_i}^{x_{i+1}} (x - m_i) f_X(x) dx = 0.} \tag{3}$$

In other words, the quantization levels are the weighted centroids of the source $x[n]$ over the interval $[x_i, x_{i+1}]$. Taking the partial derivative with respect to the quantization interval boundaries, i.e., $\{x_i,\ i = 2, 3, \dots, Q\}$ and using the derivative of an integral rule we obtain:

$$\boxed{\frac{\partial \sigma_e^2}{\partial x_j} = 0 \iff x_j = \frac{m_j + m_{j-1}}{2}.} \tag{4}$$

In other words the j^{th} quantizer boundary is just the arithmetic mean of the $(j-1)^{\text{th}}$ and j^{th} quantizer output levels. These two equations in the unknowns are however, coupled and need to be solved recursively. These equations are initialized with a choice of m_1 and $x_1 = -\infty$. The solution so obtained has to be consistent, i.e.,

$$\boxed{\int_{x_Q}^{\infty} (x - m_Q) f_X(x) dx = 0.} \tag{5}$$

If not our choice of initial conditions is inappropriate and we need to go back and feed in a different m_1 and repeat the procedure again.

Non Uniform quantization algorithm

Summarizing the results above:

1. Input a choice for m_1 and assume $x_1 = -\infty$.
2. Use Eq. (3) and Eq. (4) to obtain the next quantizer level and interval boundary.
3. Check the consistency of the solution so obtained using Eq. (5).
4. If solution is consistent then stop the algorithm. If not go back to step 1 and begin procedure again.