Minimum Phase and Allpass Systems

Minimum Phase Systems

A system function $H(z)$ is said to be a minimum phase system if all of its poles and zeros are within the unit circle. Consider a causal and stable LTI system with a difference equation representation of the form:

$$\sum_{k=0}^{N} a[k]y[n-k] \;=\; \sum_{k=0}^{M} b[k]x[n-k],$$

the system function $H(z)$ of this LTI system takes the form

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b[k]z^{-k}}{\sum_{k=0}^{N} a[k]z^{-k}}.$$

In pole-zero format this system function is given by:

$$H(z) \;=\; \left(\frac{b_o}{a_o}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}, \quad |z| > \max_{\forall i} |d_i|.$$

The minimum phase condition on the system function can then be translated into the pole zero parameters as:

$$\max_{\forall i} |c_i| < 1 \quad \& \quad \max_{\forall i} |d_i| < 1.$$

The implication of this statement is that inverse system with system function $H_i(z)$ also has it poles and zeros within the unit circle, i.e., $H_i(z)$ is also a minimum phase system:

$$H_i(z) \;=\; \left(\frac{a_o}{b_o}\right)\frac{\prod_{k=1}^{M}(1-d_k z^{-1})}{\prod_{k=1}^{N}(1-c_k z^{-1})}, \quad |z| > \max_{\forall i} |c_i|.$$

This means that the minimum-phase system is a system where both the system and its inverse are causal and stable.

Allpass systems

An allpass system is a system whose frequency response magnitude is constant for all frequencies, i.e.,

$$|H(e^{j\omega})| = c, \quad \omega \in [-\pi, \pi].$$

The general form of a implementable allpass system is:

$$H_{\mathrm{ap}}(z) = c \prod_{k=1}^{N_r} \frac{z^{-1} - p_k}{1 - p_k z^{-1}} \prod_{k=1}^{N_c} \frac{(z^{-1} - q_k)(z^{-1} - q_k^*)}{(1 - q_k z^{-1})(1 - q_k^* z^{-1})}.$$

Specifically let us look at the causal and stable LTI system with system function $H(z)$ given by:

$$H(z) \quad = \quad \frac{z^{-1} - a^*}{1 - a z^{-1}} \quad = \quad \frac{z^{-1} - r e^{-j\theta}}{1 - r e^{j\theta} z^{-1}}.$$

The corresponding frequency response for this system is:

$$H(e^{j\omega}) = e^{-j\omega} \left(\frac{1 - r e^{-j\theta} e^{j\omega}}{1 - r e^{j\theta} e^{-j\omega}} \right).$$

The magnitude response of this system is given by:

$$|H(e^{j\omega})| \quad = \quad \left| \frac{1 - r e^{-j\theta} e^{j\omega}}{1 - r e^{j\theta} e^{-j\omega}} \right| = 1.$$

Consequently this system is an all-pass system of first order. The phase response of this system is given by:

$$\mathrm{ARG}(H(e^{j\omega})) \quad = \quad -\omega - 2\tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r\cos(\omega - \theta)} \right), \quad \omega \in [-\pi, \pi].$$

The group delay of this system obtained after taking the derivative of the expression above is given by:

$$\mathrm{grd}(H(e^{j\omega})) \quad = \quad \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)}, \quad \omega \in [-\pi, \pi].$$

From the expression above we can see that the allpass system is a positive group delay system, i.e.,

$$\mathrm{grd}(H(e^{j\omega})) > 0, \quad r < 1.$$

It is easily seen that this implies that the continuous phase is negative over $\omega \in [0, \pi]$, i.e.,

$$\mathrm{Arg}(H(e^{j\omega})) < 0, \quad \omega \in [0, \pi].$$

System Function Factorization

Consider a stable and causal LTI system with system function $H(z)$. Since the system is a stable system the poles of the system function are required to be inside the *unit circle* (UC). The zeroes are however, free to wander outside. Let us assume that there are q zeroes of $H(z)$ that are outside the UC. Then the system function $H(z)$ can be rewritten as:

$$H(z) = H_1(z) \prod_{k=1}^{q} (z^{-1} - p_k), \quad |p_k| < 1.$$

Specifically the system function $H_1(z)$ is a minimum phase system function because it has all its poles and zeroes inside the UC. Rewriting this expression to include the allpass factor we obtain:

$$H(z) = \underbrace{\left(H_1(z) \prod_{k=1}^{q} (1 - p_k^* z^{-1}) \right)}_{H_{\min}(z)} \overbrace{\prod_{k=1}^{q} \left(\frac{z^{-1} - p_k}{1 - p_k^* z^{-1}} \right)}^{H_{\text{ap}}(z)}.$$

This implies that system function $H(z)$ can then be factorized into two parts as:

$$H(z) \;=\; H_{\min}(z) H_{\text{ap}}(z),$$

where $H_{\min}(z)$ is defined as the minimum phase component of $H(z)$ and $H_{\text{ap}}(z)$ is defined as the allpass part or component of $H(z)$.

Implications of the Factorization

In terms of the magnitude response $|H(e^{j\omega})|$ this factorization implies that:

$$|H(e^{j\omega})| \;=\; |H_{\min}(e^{j\omega})|.$$

In other words the magnitude of the frequency response of the system and the magnitude of the frequency response of the minimum phase part are identical. In terms of the phase response this factorization implies that:

$$\text{Arg}(H(e^{j\omega})) = \text{Arg}(H_{\min}(e^{j\omega})) + \text{Arg}(H_{\text{ap}}(e^{j\omega})).$$

This means that the minimum phase system is also a *minimum phase lag* system. In terms of group delay response this factorization implies:

$$\text{grd}(H(e^{j\omega})) = \text{grd}(H_{\min}(e^{j\omega})) + \text{grd}(H_{\text{ap}}(e^{j\omega})).$$

Using the positive group delay property of the allpass part we can infer that the minimum phase system is also a *minimum group delay* system.

Factorization Algorithm

The algorithm for obtaining the components is :

1. Assign the poles and the zeroes of $H(z)$ inside the UC to the minimum phase part $H_{\min}(z)$.
2. Reflect the zeroes of $H(z)$ that are outside the UC to their conjugate reciprocal locations.
3. The allpass part will be composed of the zeroes that lie outside the UC and the poles at the complex reciprocal locations needed to compensate for the reflection of the zeroes to the complex reciprocal locations.

Magnitude Square Factorization

Consider a system function $C(z)$ that is positive, real and analytic on the UC. Since the system function $C(z)$ is positive, real and analytic on the UC, the corresponding frequency response can be expressed in the form:

$$C(e^{j\omega}) = K|H(e^{j\omega})|^2, \quad \omega \in [-\pi, \pi] \quad K > 0.$$

The corresponding quantity system function in the Z-domain is:

$$C(z) = KH(z)H^*\left(\frac{1}{z^*}\right), \quad z_l < 1 \leq |z| < z_r.$$

This mean that the poles and zeroes of $C(z)$ come in conjugate reciprocal pairs. We can then split the poles and zeroes into two groups, one group inside the UC and the other group outside the UC as described by:

$$C(z) = K \left(\overbrace{\frac{\prod_{k=1}^{n_a} 1 - r_k z^{-1}}{\prod_{k=1}^{n_b} 1 - s_k z^{-1}}}^{H_{\min}(z)} \right) \left(\underbrace{\frac{\prod_{k=1}^{n_a} 1 - r_k^* z}{\prod_{k=1}^{n_b} 1 - s_k^* z}}_{H_{\max}(z)} \right).$$

Then the system function $C(z)$ can be factorized in the form:

$$C(z) = KH_{\min}(z)H_{\max}(z), \; K > 0,$$

where $H_{\min}(z)$ is the minimum phase part or component of $C(z)$ and

$$H_{\max}(z) = H^*_{\min}\left(\frac{1}{z^*}\right)$$

is the maximum phase part of $C(z)$. The corresponding in the time domain is:

$$c[n] = Kh_{\min}[n] * h^*_{\min}[-n].$$