Frequency Response of FIR Linear Phase Systems

FIR, linear phase systems fall into one of 4 categories:

1. M even, $h[n]$ is symmetric (Type I),
2. M odd, $h[n]$ is symmetric (Type II),
3. M even, $h[n]$ is antisymmetric (Type III),
4. M odd, $h[n]$ is antisymmetric (type IV).

In this section, will examine what these symmetry conditions translate to in terms of the system frequency response.

**Type I Systems**

FIR linear phase systems that fall into this category have an even order, i.e., $M$ is even and the impulse response of these systems is symmetric, i.e.,

$$h[n] = h[M - n], \quad n = \left(\frac{M}{2}\right) + 1, \ldots, M.$$  

The frequency response of systems in this category can be split up into 3 terms:

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h[n] \exp(-j\omega n) + \sum_{n=M/2+1}^{M} h[M - n] \exp(-j\omega n) + h\left(\frac{M}{2}\right) e^{-j\omega \frac{M}{2}}.$$  

After a substitution of variables in the second term we obtain:

$$H(e^{j\omega}) = \sum_{n=0}^{M/2-1} h[n] \exp(-j\omega n) + \sum_{n=0}^{M/2-1} h[n] \exp(j\omega n) e^{-j\omega M} + h\left[\frac{M}{2}\right] e^{-j\omega \frac{M}{2}}.$$  

Combining the first and the second sum in the frequency response and using the Euler identity we have:

$$H(e^{j\omega}) = \sum_{n=0}^{M/2-1} 2h[n] \cos(\omega n) e^{-j\omega \frac{M}{2}} + h\left[\frac{M}{2}\right] e^{-j\omega \frac{M}{2}}.$$
The frequency response of the systems in this category can then be put into the general form:

\[ H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}} a[n] \cos(\omega n)e^{-j\omega \frac{M}{2}} = A(e^{j\omega})e^{-j\omega \frac{M}{2}}, \]

where the coefficients \( a[n] \) are given by:

\[
a[n] = \begin{cases} 
  h \left( \frac{M}{2} \right) & n = 0 \\
  2h \left( \frac{M}{2} - n \right) & n = 1, \ldots, \frac{M}{2}
\end{cases}
\]

**Type II Systems**

For systems in this category, the filter order \( M \) is odd and the impulse response is symmetric. As in the previous case, we can expand the frequency response in the following fashion:

\[ H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{n=\frac{M+1}{2}}^{M} h[n]e^{-j\omega n}. \]

After a substitution of variables in the second term we obtain:

\[ H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{j\omega n}e^{-j\omega M}. \]

Combining the two sums in the frequency response and using the Euler identity we obtain:

\[ H(e^{j\omega}) = \sum_{n=0}^{\frac{M+1}{2}} 2h[n] \cos(\omega n)e^{-j\omega \frac{M}{2}}. \]

The frequency response of systems in this category can then be put in the general form of:

\[ H(e^{j\omega}) = \sum_{k=1}^{\frac{M+1}{2}} b[k] \cos \left[ \omega \left( k - \frac{1}{2} \right) \right] e^{-j\omega \frac{M}{2}}, \]

where the coefficients \( b[k] \) are given by:

\[ b[k] = 2h \left[ \frac{M + 1}{2} - k \right], \quad k = 1, \ldots, \frac{M + 1}{2}. \]
Type III Systems

The FIR linear phase systems in this category have an even model order, i.e., $M$ is even but the impulse response is antisymmetric, i.e., $h[n] = -h[M - n]$. Using the antisymmetry and the fact that $M$ is even:

$$h[M/2] = -h[-M/2] \iff h[M/2] = 0.$$

In a manner similar to the derivations for the earlier FIR linear phase systems, it can be shown that the frequency response of systems in this category have the general form:

$$H(e^{j\omega}) = j \sum_{k=1}^{M/2} c[k] \sin(k\omega) e^{-j\omega M/2},$$

where the coefficients in the sum, $c[k]$ are given by:

$$c[k] = 2h \left[ \frac{M}{2} - k \right], \quad k = 1, 2, \ldots, \frac{M}{2}.$$

Type IV Systems

For systems in this category the model order is odd, i.e., $M$ is odd but the impulse response is antisymmetric, i.e., $h[n] = -h[M - n]$. In a manner similar to the derivations for the earlier FIR linear phase systems, it can be shown that the frequency response of systems in this category have the general form:

$$H(e^{j\omega}) = j \sum_{k=1}^{M/2} d[k] \sin(k\omega) e^{-j\omega M/2} = A(e^{j\omega})e^{-j\omega M/2 + j\pi/2},$$

where the coefficients $d[k]$ in the sum are given by:

$$d[k] = 2h \left[ \frac{M + 1}{2} - k \right], \quad k = 1, 2, \ldots, \frac{M + 1}{2}.$$