Frequency Response of FIR Linear Phase Systems

FIR, linear phase systems fall into one of 4 categories:

1. M even, $h[n]$ is symmetric (Type I),
2. M odd, $h[n]$ is symmetric (Type II),
3. M even, $h[n]$ is antisymmetric (Type III),
4. M odd, $h[n]$ is antisymmetric (type IV).

In this section, will examine what these symmetry conditions translate to in terms of the system frequency response.

Type I Systems

FIR linear phase systems that fall into this category have an even order, i.e., M is even and the impulse response of these systems is symmetric, i.e.,

$$h[n] = h[M-n], \quad n = \left(\frac{M}{2}\right) + 1, \dots, M.$$

The frequency response of systems in this category can be split up into 3 terms:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}-1} h[n]\exp(-j\omega n) + \sum_{n=\frac{M}{2}+1}^{M} h[M-n]\exp(-j\omega n) + h\left(\frac{M}{2}\right)e^{-j\omega\frac{M}{2}}.$$

After a substitution of variables in the second term we obtain:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}-1} h[n]\exp(-j\omega n) + \sum_{n=0}^{\frac{M}{2}-1} h[n]\exp(j\omega n)e^{-j\omega M} + h\left[\frac{M}{2}\right]e^{-j\omega\frac{M}{2}}.$$

Combining the first and the second sum in the frequency response and using the Euler identity we have:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}-1} 2h[n]\cos(\omega n)e^{-j\omega\frac{M}{2}} + h\left[\frac{M}{2}\right]e^{-j\omega\frac{M}{2}}.$$

The frequency response of the systems in this category can then be put into the general form:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}} a[n]\cos(\omega n)e^{-j\omega\frac{M}{2}} = A(e^{j\omega})e^{-j\omega\frac{M}{2}},$$

where the coefficients $a[n]$ are given by:

$$a[n] = \begin{cases} h\left[\frac{M}{2}\right] & n = 0 \\ 2h[\frac{M}{2} - n] & n = 1, 2, \dots, \frac{M}{2} \end{cases}$$

Type II Systems

For systems in this category, the filter order M is odd and the impulse response is symmetric. As in the previous case, we can expand the frequency response in the following fashion:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{n=\frac{M+1}{2}}^{M} h[M-n]e^{-j\omega n}.$$

After a substitution of variables in the second term we obtain:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j\omega n} + \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{j\omega n}e^{-j\omega M}.$$

Combining the two sums in the frequency response and using the Euler identity we obtain:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} 2h[n]\cos(\omega n)e^{-j\omega\frac{M}{2}}.$$

The frequency response of systems in this category can then be put in the general form of:

$$H(e^{j\omega}) = \sum_{k=1}^{\frac{M+1}{2}} b[k]\cos\left[\omega\left(k - \frac{1}{2}\right)\right]e^{-j\omega\frac{M}{2}},$$

where the coefficients $b[k]$ are given by:

$$b[k] = 2h\left[\frac{M+1}{2} - k\right], \quad k = 1, 2, \dots, \frac{M+1}{2}.$$

Type III Systems

The FIR linear phase systems in this category have an even model order, i.e., M is even but the impulse response is antisymmetric, i.e., $h[n] = -h[M-n]$. Using the antisymmetry and the fact that M is even:

$$h[M/2] = -h[-M/2] \longleftrightarrow h[M/2] = 0.$$

In a manner similar to the derivations for the earlier FIR linear phase systems, it can be shown that the frequency response of systems in this category have the general form:

$$H(e^{j\omega}) = j \sum_{k=1}^{\frac{M}{2}} c[k] \sin(k\omega) e^{-j\omega \frac{M}{2}},$$

where the coefficients in the sum, $c[k]$ are given by:

$$c[k] = 2h\left[\frac{M}{2} - k\right], \quad k = 1, 2, \dots, \frac{M}{2}.$$

Type IV Systems

For systems in this category the model order is odd, i.e., M is odd but the impulse response is antisymmetric, i.e., $h[n] = -h[M-n]$. In a manner similar to the derivations for the earlier FIR linear phase systems, it can be shown that the frequency response of systems in this category have the general form:

$$H(e^{j\omega}) = j \sum_{k=1}^{\frac{M}{2}} d[k] \sin(k\omega) e^{-j\omega \frac{M}{2}} = A(e^{j\omega}) e^{-j\omega \frac{M}{2} + j\frac{\pi}{2}},$$

where the coefficients $d[k]$ in the sum are given by:

$$d[k] = 2h\left[\frac{M+1}{2} - k\right], \quad k = 1, 2, \dots, \frac{M+1}{2}.$$

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