Quantization Noise Shaping Via Oversampling

Now that we have a basic idea of the effect that a decimation system has on a random input signal, we can analyze a noise shaping systems that are based on oversampling of a analog signal $x_c(t)$ by a factor of L. Assuming first that B is sufficiently large and that the quantizer's maximum output level X_m is matched to the dynamic range of signal we can apply the additive noise model for the uniform quantizer in these systems. First note that eventhough there are multirate operations involved, the effective system is still linear and consequently the principle of superposition still applies. The signal component of the quantized output will be denoted $x_q[n]$ and the noise component $e_q[n]$ that is assumed to be white and uniformly distributed. The SNR at the quantizer output, that will be used as a reference is given by:

$$\text{SNR}_o = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_{\max}}{\sigma_x}\right).$$

For the noise shaping system with just oversampling and filtering in it, we are simply decimating the quantized signal. The signal component of the quantizer output passes through the through without distortion. The noise component at the output of the noise shaping system is still white but has a noise variance that has been reduced by the factor L. The SNR at the output of this system is therefore:

$$\text{SNR}_1 = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_{\max}}{\sigma_x}\right) + 10\log_{10}(L).$$

Specifically when the oversampling factor is of the form $L = 2^\nu$, then this corresponds to a increase in the resolution of $x_{qo}[n]$ by a factor of $\frac{\nu}{2}$ bits. For the $\Sigma - \Delta$ system with a single feedback loop, we first look at the reduction of the quantization noise power introduced by the loop. The power spectrum of $f[n]$, i.e., the shaped quantization noise is given by:

$$P_{ff}(e^{j\omega}) = |1 - e^{-j\omega}|^2 P_{ee}(e^{j\omega})$$

After the LPF operation and downsampling the average power of this noise becomes:

$$P^o_{\text{ave}} = \frac{2\sigma_e^2}{\pi T_s}\left(\frac{\pi}{L} - \sin\frac{\pi}{L}\right).$$

If we now employ a Taylor series expansion for the second term in the expression for large L and retain just two terms in the series we obtain:

$$P^o_{\text{ave}} = \frac{2\sigma_e^2}{\pi T_s}\left(\frac{\pi^3}{6L^3}\right) = \frac{\sigma_e^2}{T_s}\frac{\pi^2}{3L^3}.$$

The SNR at the output of this noise shaping system is then expressed as:

$$\text{SNR}_2 = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_{\max}}{\sigma_x}\right) + 10\log_{10}\left(\frac{3L^3}{\pi^2}\right)$$

If we used a factor of $L = 4$, this would correspond to an SNR increase of 12.89 dB which corresponds to a increase in the resolution of the quantizer by a factor of 2.14 bits. Now consider a third noise shaping system where we have 2 feedback loops instead of just one. The power spectrum of the noise at the output of the feedback loop is:

$$P_{ff}(e^{j\omega}) = |1 - e^{-j\omega}|^4 P_{ee}(e^{j\omega}) = 16\sin^4\left(\frac{\omega}{2}\right)\frac{\sigma_e^2}{T_s}$$

For small frequencies in the passband of the LPF with cut-off frequency $\omega_c = \frac{\pi}{L}$ that follows the feedback loop we can approximate this as:

$$P_{ff}(e^{j\omega}) \approx \frac{\sigma^2}{T_s}\omega^4, \quad \omega \in [-\pi, \pi].$$

The average power at the output of this noise shaping system is therefore:

$$P^o_{\text{ave}} = \frac{\sigma_e^2}{T_s}\frac{\pi^4}{5M^5}.$$

The corresponding SNR at the output of this noise shaping system is:

$$\text{SNR}_3 = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_{\max}}{\sigma_x}\right) + 10\log_{10}\left(\frac{5L^5}{\pi^4}\right)$$

Specifically for $L = 4$ this corresponds to a 17.21 dB increase in SNR and a corresponding 2.86 bit increase in resolution of the quantizer.