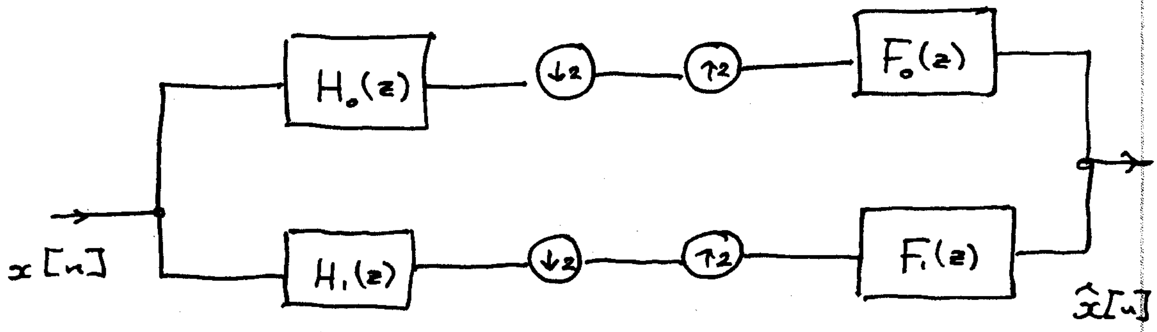
ECE-539
Spring 2005

Paraunitary Filterbanks



PR Equations:

$$\begin{pmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{pmatrix} \begin{pmatrix} F_0(z) \\ F_1(z) \end{pmatrix} = \begin{pmatrix} 2c\, z^{-n_0} \\ 0 \end{pmatrix}$$

If we adopt the paraunitary QMF solution:

$$h_1[n] = (-1)^n h_0^*[N-n]$$

$$H_1(z) = -H_0^*(-1/z^*)\, z^{-N} \qquad (1)$$

$$f_0[n] = h_0^*[N-n]$$

$$F_0(z) = H_0^*(1/z^*)\, z^{-N} \qquad (2)$$

$$f_1[n] = h_1^*[N-n]$$

$$= z^{-N} H_1^*(1/z^*) \qquad (3)$$

Substituting (1), (2) & (3) into PR equations we have

Eq for alias cancelation:

$$H_0^*\left(\frac{1}{z^*}\right) z^{-N} H_0(z) + z^{-N} H_1^*(1/z^*) H_1(-z)$$

$$= H_0^*(1/z^*) z^{-N} H_0(-z) + z^{-N} \underbrace{H_0^*(1/z^*)(-z)^{-N}}_{} \; H_0(-z)\left(\frac{1}{z}\right)^{-N}$$

$$= z^{-N} H_0(-z) H_0^*(1/z^*) \left(1 + (-1)^N\right) = 0$$

for N odd

$$\Rightarrow \boxed{H_0(-z) F_0(z) + H_1(-z) F_1(z) = 0} \quad (4)$$

$$H_0(z) F_0(z) = H_0(z) H_0^*(1/z^*) z^{-N}$$
$$H_1(z) F_1(z) = H_1(z) H_1^*(1/z^*) z^{-N}$$

$$H_0(z) F_0(z) + H_1(z) F_1(z) = z^{-N}\left(H_0(z) H_0^*(1/z^*) + H_1(z) H_1^*(1/z^*)\right)$$

$$H_1(z) H_1^*(1/z^*) = -\left(H_0^*(-1/z^*) z^{-N}\right) \cdot -\left(H_0(-z)\left(\frac{1}{z}\right)^{-N}\right)$$

$$= H_0(-z) H_0^*\left(-\frac{1}{z^*}\right)$$

$\Rightarrow$ If we design $H_0(z) \ni$

$$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega-\pi)})|^2 = c$$

$$\Rightarrow \boxed{H_0(z) F_0(z) + H_1(z) F_1(z) = c z^{-N}} \quad (5)$$

Paraunitary QMF solution on UC

$$H_1(z) = (-1)\, H_0^*\left(-\frac{1}{z^*}\right) z^{-N}$$

$$H_1(e^{j\omega}) = (-1)\, H_0^*(-e^{j\omega})\, e^{-j\omega N}$$

$$|H_1(e^{j\omega})| = |H_0^*(-e^{j\omega})|$$

$$\boxed{|H_1(e^{j\omega})| = |H_0(e^{j(\omega-\pi)})|}$$

(Quadrature Mirror Symmetry)

$$F_0(e^{j\omega}) = H_0^*(e^{j\omega})\, e^{-j\omega N}$$

$$|F_0(e^{j\omega})| = |H_0(e^{j\omega})|$$

Similarly

$$|F_1(e^{j\omega})| = |H_1(e^{j\omega})|$$

$\Rightarrow$ Eventhough $H_1(z) \neq H_0(-z)$ QMF property still holds

$\Rightarrow$ Solution is referred to as paraunitary QMF solution

Paraunitary property:

If $H_0(z)$ is designed $\ni$

$$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega-\pi)})|^2 = c \quad (5)$$

(power - comp)
PC

$$\Longrightarrow \quad |H_1(e^{j\omega})|^2 + |H_1(e^{j(\omega-\pi)})|^2$$

$$= |H_0(e^{j(\omega-\pi)})|^2 + |H_0(e^{j\omega})|^2 = c \quad (6)$$

$\Longrightarrow$ $H_1(e^{j\omega})$ is also power - complementary

Also from solution:

$$H_0^*\left(\tfrac{1}{z^*}\right) H_1(z) + H_1(-z)\, H_0^*(-1/z^*)$$

$$= H_0^*(1/z^*)\,(-1)\, z^{-N} H_0^*(-1/z^*) + H_0^*(-1/z^*)\,(-1)\,(-z)^{-N} H_0^*(1/z^*)$$

$$= -z^{-N} H_0^*\left(\tfrac{1}{z^*}\right) H_0^*(-1/z^*)\,\left(1 + (-1)^N\right)$$

$$= 0, \quad N \text{ odd} \quad (7)$$

Similarly we can show

$$H_1^*(1/z^*)\, H_0(z) + H_0(-z)\, H_1^*\left(-\tfrac{1}{z^*}\right) = 0, \quad N \text{ odd} \quad (8)$$

Combining PC properties

$$\begin{bmatrix} H_0^*(1/z^*) & H_0^*(-1/z^*) \\ H_1(1/z^*) & H_1^*(-1/z^*) \end{bmatrix} \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}$$

$$= \quad c\, I_{2\times 2} \; , \quad N \text{ odd}$$

(Paraunitary Property)