Examples: Polyphase Decomposition

Consider a *moving average* system with system function of the form:

$$H(z) = 1 - 0.5z^{-1} + 0.25z^{-2} - 0.125z^{-3} + 0.0625z^{-4}$$

The type I polyphase components with respect to $M = 2$ obtained by grouping the terms into two sets is given by:

$$E_o(z) = 1 + 0.25z^{-1} + 0.0625z^{-2}, \quad E_1(z) = -0.5 - 0.25z^{-1}.$$

Now consider a first-order *auto regressive* (AR) stable and causal system with system function $H(z)$ given by:

$$H(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > \alpha, \quad \alpha < 1.$$

A power series expansion of $H(z)$ using the geometric series expansion is given by:

$$H(z) = 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} + \alpha^4 z^{-4} \ldots$$

Grouping the terms into two sets yields the type I polyphase components with respect to $M = 2$:

$$\begin{aligned} E_o(z) &= \frac{1}{1 - \alpha^2 z^{-1}} \\ E_1(z) &= \frac{\alpha}{1 - \alpha^2 z^{-1}}. \end{aligned}$$

In this case the system function $H(z)$ is expanded as:

$$H(z) = E_o(z^2) + z^{-1}E_1(z^2).$$

As a first observation note that the ROC of these polyphase filters $E_i(z)$, i.e., $|z| > \alpha^2$ is different from the ROC of $H(z)$. If however, we group the terms with respect to the integer $M = 3$ we obtain a different set of polyphase components:

$$\begin{aligned} E_o(z) &= \frac{1}{1 - \alpha^3 z^{-1}} \\ E_1(z) &= \frac{\alpha}{1 - \alpha^3 z^{-1}} \\ E_2(z) &= \frac{\alpha^2}{1 - \alpha^3 z^{-1}}. \end{aligned}$$

For this case, the system function $H(z)$ is expanded as:

$$H(z) = E_o(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3).$$

The corresponding type II polyphase expansion of $H(z)$ would be:

$$H(z) = R_2(z^3) + z^{-1}R_1(z^3) + z^{-2}R_o(z^3).$$

Let us now look at an example with a second–order system specifically a digital resonator with system function:

$$H(z) = \frac{1}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}}, \quad |z| > R, \quad R < 1.$$

If we factorize the denominator into two parts the system function can be rewritten as:

$$H(z) = \frac{1}{(1 - Re^{j\theta}z^{-1})(1 - Re^{-j\theta}z^{-1})}.$$

If we multiply numerator by the factor $(1+Re^{j\theta}z^{-1})(1+Re^{-j\theta}z^{-1})$ and simplify the resultant expression we obtain the following type I polyphase components:

$$E_o(z) = \frac{1 + R^2 z^{-1}}{1 - 2R^2\cos 2\theta z^{-1} + R^4 z^{-2}}, \quad E_1(z) = \frac{2R\cos\theta}{1 - 2R^2\cos 2\theta z^{-1} + R^4 z^{-2}}$$