Examples: Polyphase Decomposition

Consider a moving average system with system function of the form:

\[ H(z) = 1 - 0.5z^{-1} + 0.25z^{-2} - 0.125z^{-3} + 0.0625z^{-4} \]

The type I polyphase components with respect to \( M = 2 \) obtained by grouping the terms into two sets is given by:

\[ E_0(z) = 1 + 0.25z^{-1} + 0.0625z^{-2}, \quad E_1(z) = -0.5 - 0.25z^{-1}. \]

Now consider a first-order \textit{auto regressive} (AR) stable and causal system with system function \( H(z) \) given by:

\[ H(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > \alpha, \quad \alpha < 1. \]

A power series expansion of \( H(z) \) using the geometric series expansion is given by:

\[ H(z) = 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} + \alpha^4 z^{-4} \ldots \]

Grouping the terms into two sets yields the type I polyphase components with respect to \( M = 2 \):

\[ E_0(z) = \frac{1}{1 - \alpha^2 z^{-1}}, \quad E_1(z) = \frac{\alpha}{1 - \alpha^2 z^{-1}}. \]

In this case the system function \( H(z) \) is expanded as:

\[ H(z) = E_0(z^2) + z^{-1} E_1(z^2). \]

As a first observation note that the ROC of these polyphase filters \( E_i(z) \), i.e., \(|z| > \alpha^2\) is different from the ROC of \( H(z) \). If however, we group the terms with respect to the integer \( M = 3 \) we obtain a different set of polyphase components:

\[ E_0(z) = \frac{1}{1 - \alpha^3 z^{-1}}, \quad E_1(z) = \frac{\alpha}{1 - \alpha^3 z^{-1}} \]

\[ E_2(z) = \frac{\alpha^2}{1 - \alpha^3 z^{-1}}. \]

For this case, the system function \( H(z) \) is expanded as:

\[ H(z) = E_0(z^3) + z^{-1} E_1(z^3) + z^{-2} E_2(z^3). \]
The corresponding type II polyphase expansion of $H(z)$ would be:

$$H(z) = R_2(z^3) + z^{-1}R_1(z^3) + z^{-2}R_o(z^3).$$

Let us now look at an example with a second-order system specifically a digital resonator with system function:

$$H(z) = \frac{1}{1 - 2R \cos \theta z^{-1} + R^2 z^{-2}}, \quad |z| > R, \quad R < 1.$$ 

If we factorize the denominator into two parts the system function can be rewritten as:

$$H(z) = \frac{1}{(1 - Re^{j\theta} z^{-1})(1 - Re^{-j\theta} z^{-1})}.$$ 

If we multiply numerator by the factor $(1 + Re^{j\theta} z^{-1})(1 + Re^{-j\theta} z^{-1})$ and simplify the resultant expression we obtain the following type I polyphase components:

$$E_o(z) = \frac{1 + R^2 z^{-1}}{1 - 2R^2 \cos 2\theta z^{-1} + R^4 z^{-2}}, \quad E_1(z) = \frac{2R \cos \theta}{1 - 2R^2 \cos 2\theta z^{-1} + R^4 z^{-2}}.$$