Example: Power Spectral Factorization

Consider a zero-mean, second-order, WSS, random sequence \( x[n] \), whose power spectral density has the form:

\[
P_{xx}(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = 12.5 - 10 \cos \omega \quad \frac{1.64 - 1.6 \cos \omega}{1.64 - 0.8 \cos \omega}, \quad \omega \in [-\pi, \pi].
\]

First note that the maximum and minimum values of the numerator and denominator are positive:

\[
B_{\text{max}} = 22.5, \quad B_{\text{min}} = 2.5, \quad A_{\text{max}} = 3.24, \quad A_{\text{min}} = 0.04.
\]

Furthermore both the numerator and denominator of the PSD are purely real. This PSD therefore satisfies the conditions set by the factorization theorem.

Using the Euler identity and replacing \( e^{j\omega} \) by \( z \) yields:

\[
P_{xx}(z) = 10 \left( 1 - 0.5z^{-1} \right) \left( 1 - 0.5z \right) \quad 0.8 < |z| < 1.25.
\]

Grouping the causal terms and the non causal terms we have:

\[
H_{\text{min}}(z) = \frac{1 - 0.5z^{-1}}{1 - 0.8z^{-1}}, \quad |z| > 0.8, \quad H_{\text{max}}(z) = \frac{1 - 0.5z}{1 - 0.8z}, \quad |z| < 1.25, \quad \sigma_v^2 = 10.
\]

It can indeed be verified that \( H_{\text{min}}(z) \) corresponds to a monic, minimum phase system function and \( H_{\text{max}}(z) \) corresponds to maximum phase system function. The average power of this random sequence can be obtained from the factorization using:

\[
r_{xx}[0] = \text{coefficient on } z^0 \text{ in } P_{xx}(z) = 12.5.
\]

Now consider the causal system function of the form:

\[
H_{\text{white}}(z) = \frac{1}{\sigma_v H_{\text{min}}(z)}, \quad |z| > 0.5.
\]

If the random sequence \( x[n] \) is the input signal to this system, the power spectrum of the output random sequence is given by:

\[
P_{yy}(z) = P_{xx}(z)H(z)H^* \left( \frac{1}{z^*} \right) = P_{xx}(z) \left( \frac{1}{\sigma_v H(z)} \right) \left( \frac{1}{\sigma_v H^*(1/z^*)} \right).
\]

Utilizing the PSD factorization it is easy to see that the numerator is just the same as the denominator and consequently \( P_{yy}(z) = 1 \). This implies that \( H_{\text{white}}(z) \) corresponds to the system function of a whitening system that takes the sequence \( x[n] \) and converts it into zero-mean, unit variance white noise (weak sense).