Example: Power Spectral Factorization

Consider a zero-mean, second-order, WSS, random sequence $x[n]$, whose power spectral density has the form:

$$P_{xx}(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = \frac{12.5 - 10\cos\omega}{1.64 - 1.6\cos\omega}, \quad \omega \in [-\pi, \pi].$$

First note that the maximum and minimum values of the numerator and denominator are positive:

$$B_{\max} = 22.5, \quad B_{\min} = 2.5, \quad A_{\max} = 3.24, \quad A_{\min} = 0.04.$$

Furthermore both the numerator and denominator of the PSD are purely real. This PSD therefore satisfies the conditions set by the factorization theorem. Using the Euler identity and replacing $e^{j\omega}$ by z yields:

$$P_{xx}(z) = \frac{12.5 - 5z - 5z^{-1}}{1.64 - 0.8z^{-1} - 0.8z} = 10\left(\frac{1.25 - 0.5z - 0.5z^{-1}}{1.64 - 0.8z^{-1} - 0.8z}\right).$$

Factorization of both the numerator and denominator yields:

$$P_{xx}(z) = 10\frac{(1 - 0.5z^{-1})(1 - 0.5z)}{(1 - 0.8z^{-1})(1 - 0.8z)}, \quad 0.8 < |z| < 1.25.$$

Grouping the causal terms and the non causal terms we have:

$$H_{\min}(z) = \frac{1 - 0.5z^{-1}}{1 - 0.8z^{-1}}, \quad |z| > 0.8, \quad H_{\max}(z) = \frac{1 - 0.5z}{1 - 0.8z}, \quad |z| < 1.25, \quad \sigma_v^2 = 10.$$

It can indeed be verified that $H_{\min}(z)$ corresponds to a monic, minimum phase system function and $H_{\max}(z)$ corresponds to maximum phase system function. The average power of this random sequence can be obtained from the factorization using:

$$r_{xx}[0] = \text{coefficient on } z^0 \text{ in } P_{xx}(z) = 12.5.$$

Now consider the causal system function of the form:

$$H_{\text{white}}(z) = \frac{1}{\sigma_v H_{\min}(z)}, \quad |z| > 0.5.$$

If the random sequence $x[n]$ is the input signal to this system, the power spectrum of the output random sequence is given by:

$$P_{yy}(z) = P_{xx}(z)H(z)H^*\left(\frac{1}{z^*}\right) = P_{xx}(z)\left(\frac{1}{\sigma_v H(z)}\right)\left(\frac{1}{\sigma_v H^*(1/z^*)}\right).$$

Utilizing the PSD factorization it is easy to see that the numerator is just the same as the denominator and consequently $P_{yy}(z) = 1$. This implies that $H_{\text{white}}(z)$ corresponds to the system function of a whitening system that takes the sequence $x[n]$ and converts it into zero-mean, unit variance white noise (weak sense).