Radix 2 algorithms ($N = 2^{\gamma}$)

Decimation in Time (DIT)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}, \quad 0 \le k \le N-1$$

$$X[k] = \sum_{\substack{n=0 \\ n \text{ even}}}^{N-2} x[n] W_N^{nk} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{N-1} x[n] W_N^{nk}$$

↓ Substitute $n = 2r$ (even terms)

↓ Substitute $n = 2r+1$ (odd terms)

$$X[k] = \sum_{r=0}^{N/2-1} x[2r] W_N^{2rk} + \sum_{r=0}^{N/2-1} x[2r+1] W_N^{(2r+1)k}$$

$$X[k] = \underbrace{\sum_{r=0}^{N/2-1} x[2r] W_{N/2}^{rk}}_{G[k]} + W_N^{k} \underbrace{\sum_{r=0}^{N/2-1} x[2r+1] W_{N/2}^{rk}}_{H[k]}$$

$$X[k] = G[k] + W_N^{k} H[k], \quad 0 \le k \le N-1$$

- $G[k]$ & $H[k]$ are $N/2$ pt transforms
- $G[k]$ & $H[k]$ are $N/2$ periodic sequences

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \left(\begin{array}{c|c} I_{N/2 \times N/2} & C_{N/2} \\ \hline I_{N/2 \times N/2} & -C_{N/2} \end{array}\right) \begin{pmatrix} G[0] \\ \vdots \\ G[N/2-1] \\ H[0] \\ \vdots \\ H[N/2-1] \end{pmatrix}$$

where

$$C_{N/2} \triangleq \begin{pmatrix} W_N^0 & & & 0 \\ & W_N^1 & & \\ & & \ddots & \\ 0 & & & W_N^{N/2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}}_{\underline{X}} = \begin{pmatrix} I_{N/2 \times N/2} & C_{N/2} \\ I_{N/2 \times N/2} & C_{N/2} \end{pmatrix} \begin{pmatrix} \underline{G} \\ \underline{H} \end{pmatrix}$$

Further decomposition:

$$G[k] = P[k] + W_{N/2}^{k} Q[k]$$

$$H[k] = R[k] + W_{N/2}^{k} S[k],$$

$$0 \le k \le N/2 - 1$$

- $P[k]$, $Q[k]$, $R[k]$, $S[k]$ are $N/4$ pt DFT's

$$P[k] = \sum_{r=0}^{N/4-1} x[4r]\, W_{N/4}^{rk}$$

$$Q[k] = \sum_{r=0}^{N/4-1} x[4r+2]\, W_{N/4}^{rk}$$

$$\begin{pmatrix} \underline{G} \\ \underline{H} \end{pmatrix} = \begin{pmatrix} I_{N/4} & C_{N/4} & & \\ I_{N/4} & -C_{N/4} & & 0 \\ & 0 & I_{N/4} & C_{N/4} \\ & & I_{N/4} & -C_{N/4} \end{pmatrix} \begin{pmatrix} \underline{P} \\ \underline{Q} \\ \underline{R} \\ \underline{S} \end{pmatrix}$$

Repeat this procedure untill 2 point DFT's are reached

2 point DFT:

$$\begin{pmatrix} X[0] \\ X[1] \end{pmatrix} = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{H \text{ (Walsh-Hadamard Matrix)}} \begin{pmatrix} x[0] \\ x[1] \end{pmatrix}$$

$$N = 8 = 2^3 \quad (3 \text{ stages})$$

$\underline{X} = M_8 M_4 M_2 E \underline{x}$, where

$$M_8 \triangleq \begin{pmatrix} I_4 & C_4 \\ I_4 & -C_4 \end{pmatrix} \quad \begin{pmatrix} 8 \text{ pt} \\ \text{DFT's} \end{pmatrix}$$

$$M_4 \triangleq \left(\begin{array}{cc|cc} I_2 & C_2 & & \\ I_2 & -C_2 & & 0 \\ \hline & & I_2 & C_2 \\ & 0 & I_2 & -C_2 \end{array}\right) \quad \begin{pmatrix} 4 \text{ pt} \\ \text{DFT's} \end{pmatrix}$$

$$M_2 \triangleq \begin{pmatrix} H & & & 0 \\ & H & & \\ & & H & \\ 0 & & & H \end{pmatrix} \quad \begin{pmatrix} 2 \text{ pt} \\ \text{DFT's} \end{pmatrix}$$

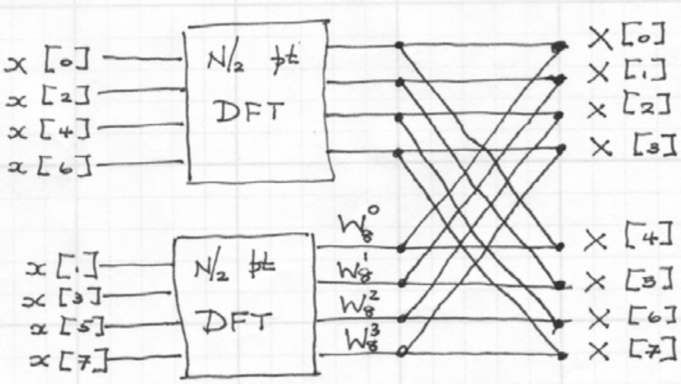
$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{Bit reversal})$$

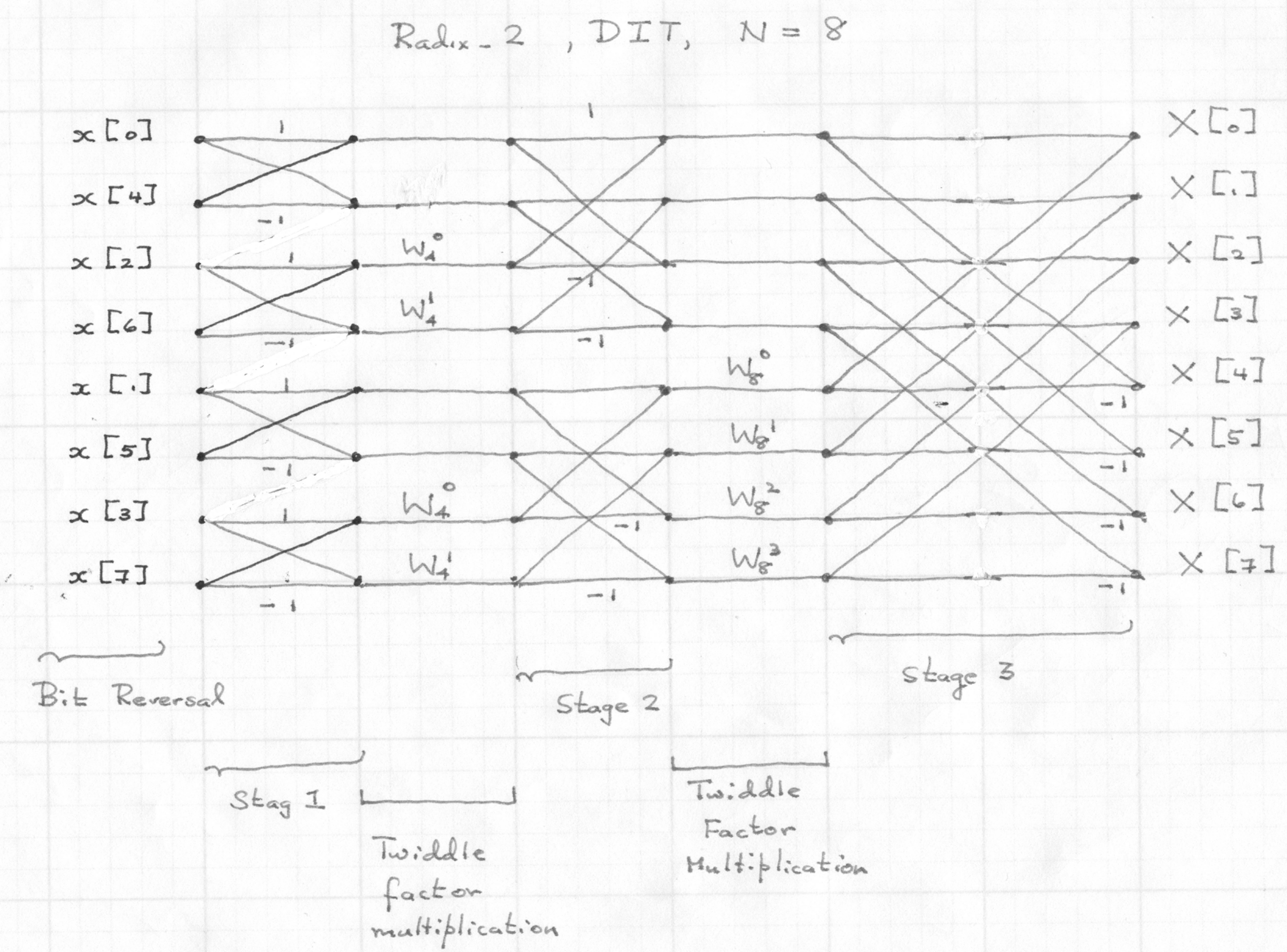
- Each of these matrices is sparse
- W_N & E are both symmetric matrices, i.e,

(1) $W_N = W_N^T \quad \left(e^{-j\frac{2\pi}{N}lm} = e^{-j\frac{2\pi}{N}ml} \right)$

(2) $E = E^T$

Flow Graph Structure




Radix-2 , DIT, N = 8
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
1
-1
W4^0
W4^1
W8^0
W8^1
W8^2
W8^3
Bit Reversal
Stag 1
Twiddle factor multiplication
Stage 2
Twiddle Factor Multiplication
Stage 3

Computational Complexity

Direct Method:

$$\mu(N) \approx N^2 \text{ complex multiplications}$$

Decimation in Time:

$$\mu(N) = \frac{N}{2} \text{ (Twiddle factors)} + 2\,\mu\left(\frac{N}{2}\right)$$

$$\mu(N) = \frac{N}{2} + 2\left(\frac{N}{4} + 2\,\mu\left(\frac{N}{4}\right)\right)$$

$$= \frac{N}{2} + \frac{N}{2} + 4\,\mu\left(\frac{N}{4}\right)$$

$$= \frac{N}{2} + \frac{N}{2} + \frac{N}{2} + 8\,\mu\left(N/4\right)$$

This decomposition can be done at most γ times, where $N = 2^\gamma$

$$\mu(N) = \frac{N}{2}\gamma = \frac{N}{2}\log_2(N)$$

- The factor $\frac{N}{2}$ for twiddle factors appears because only half of those need to be computed.
- For large N, this is a significant saving, e.g.

  $N = 1024 = 2^{10}$

  $\mu(N)$ direct : 2^{20} multiplies

  $\mu(N)$ DIT : $5\,2^{10}$ multiplies

Decimation in Frequency (DIF)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \quad , \quad 0 \le k \le N-1$$

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{2rn} \quad , \quad 0 \le r \le \frac{N}{2} - 1$$

$$X[2r] = \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{rn} + \sum_{n=N/2}^{N-1} x[n] W_{N/2}^{rn}$$

$$X[2r] = \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{nr} + \sum_{n=0}^{N/2-1} x[n+\tfrac{N}{2}] W_{N/2}^{(n+N/2)r}$$

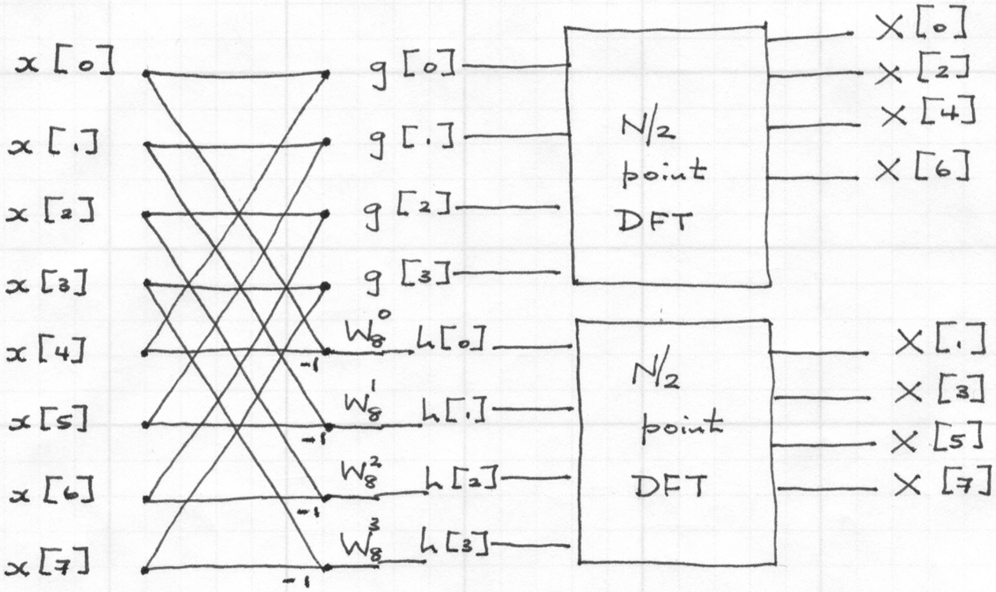
$$X[2r] = \sum_{n=0}^{N/2-1} \underbrace{\left(x[n] + x[n+N/2]\right)}_{g[n]} W_{N/2}^{nr}$$

$$X[2r+1] = \sum_{n=0}^{N/2-1} x[n] W_N^{(2r+1)n} + \sum_{N/2}^{N-1} x[n] W_N^{(2r+1)n}$$

$$X[2r+1] = \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{(r+1/2)n} + \sum_{n=0}^{N/2-1} x[n+N/2] W_{N/2}^{(r+1/2)(n+N/2)}$$

$$X[2r+1] = \sum_{n=0}^{N/2-1} \underbrace{\left(x[n] - x[n+\tfrac{N}{2}]\right) W_N^{n}}_{h[n]} \quad W_{N/2}^{nr}$$

Signal Flow Graph:



Further Decomposition:

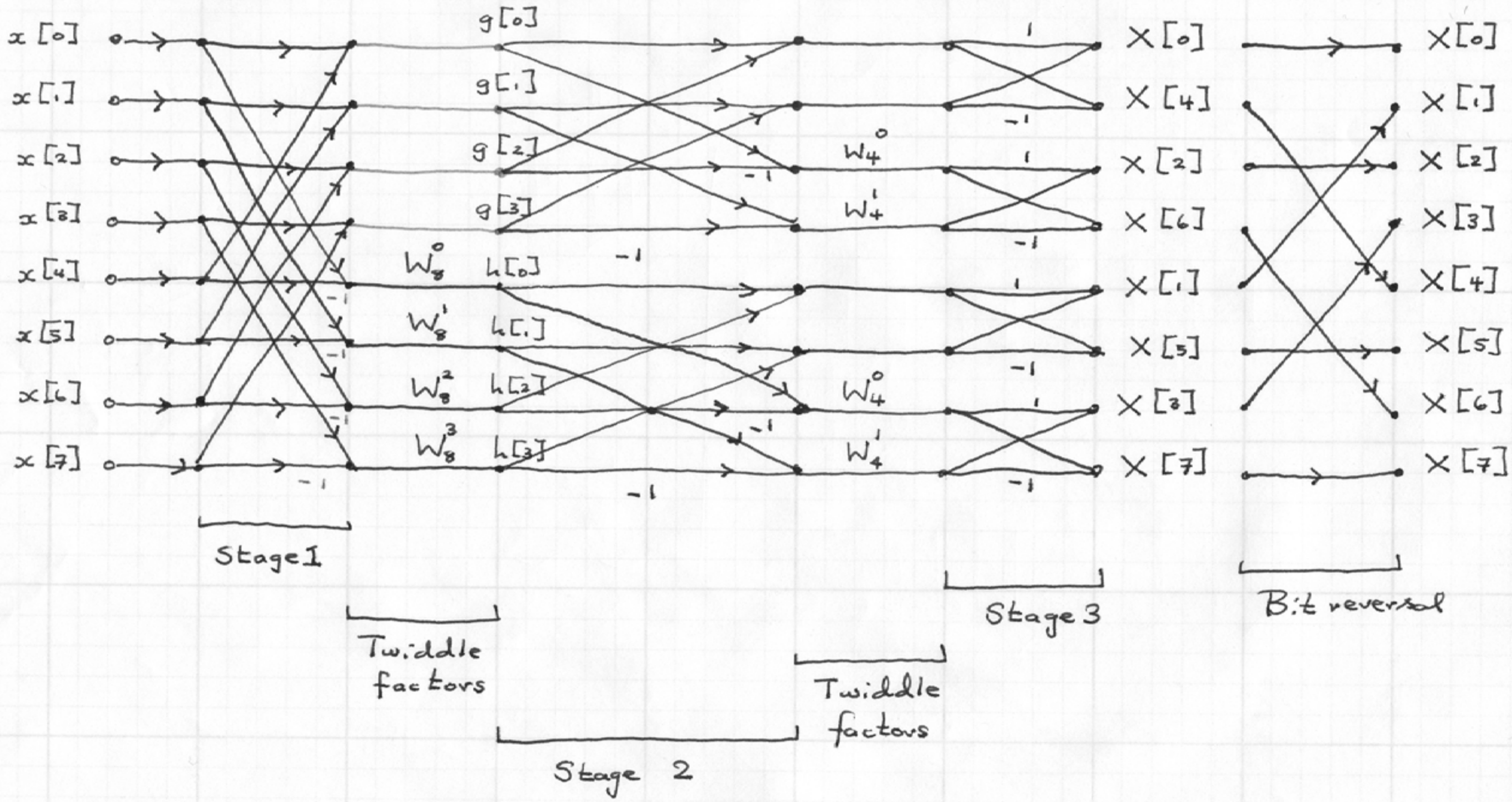
$$X[4p] = \sum_{n=0}^{N/4-1} \left(g[n] + g[n+N/4]\right) W_{N/4}^{np}$$

$$X[4p+2] = \sum_{n=0}^{N/4-1} \left(g[n] - g[n+N/4]\right) W_{N/2}^{n} W_{N/4}^{np}$$

$$X[4p+1] = \sum_{n=0}^{N/4-1} \left(h[n] + h[n+N/4]\right) W_{N/4}^{np}$$

$$X[4p+3] = \sum_{n=0}^{N/4-1} \left(h[n] - h[n+N/4]\right) W_{N/2}^{n} W_{N/4}^{np}$$

Decimation in Frequency (DIF), N = 8



Matrix Formulation:

$$\text{DIT:} \quad \underline{X} = \underbrace{M_N M_{N/2} M_{N/4} \cdots M_2 E_N}_{W_N} \underline{x}$$

$$W_N = M_N M_{N/2} M_{N/4} \cdots\cdots M_2 E_N$$

$$W_N^T = W_N = E_N^T M_2^T M_4^T \cdots M_{N/4}^T M_{N/2}^T M_N^T$$

$$\text{Also} \quad E_N^T = E_N$$

$$\Longrightarrow W_N = E_N M_2^T M_4^T \cdots M_{N/4}^T M_{N/2}^T M_N^T$$

$$\Longrightarrow \underline{X} = E_N M_2^T M_4^T \cdots M_{N/4}^T M_{N/2}^T M_N^T \underline{x}$$

- Decimation in Frequency (DIF) corresponds to the transpose form of Decimation in time

- Both algorithms have a complexity $\mu(N) = \frac{N}{2} \log_2(N)$ complex multiplies