ECE-539, SPRING 2012
DIGITAL SIGNAL PROCESSING

RELATION BETWEEN ℓ^1, ℓ^2, ℓ^∞

NORM INPUT SCALING:

$$\left(\sum_{n=-\infty}^{\infty} |h[n]|\right)^2 = \left(\sum_{p=-\infty}^{\infty} |h[p]|\right)\left(\sum_{q=-\infty}^{\infty} |h[q]|\right)$$

$$= \sum_{p=-\infty}^{\infty} |h[p]|^2 + \underbrace{\sum_{p=-\infty}^{\infty} \sum_{\substack{q=-\infty \\ p\neq q}}^{\infty} |h[p]|\,|h[q]|}_{>0}$$

$$\|h[n]\|_1^2 \geq \|h[n]\|_2^2$$

or

$$\|h[n]\|_1 > \|h[n]\|_2 \qquad \text{(a)}$$

$$\|h[n]\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega$$

$$\leq \max_{\omega} |H(e^{j\omega})|^2$$

$$\|H(e^{j\omega})\|_2^2 \leq \max_{\omega} |H(e^{j\omega})|^2$$

or $$\|H(e^{j\omega})\|_2 < \|H(e^{j\omega})\|_\infty \qquad \text{(b)}$$

Furthermore:

$$\| H(e^{j\omega}) \|_\infty = \max_{\omega \in [-\pi, \pi]} |H(e^{j\omega})|$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \exp(-j\omega n)$$

$$|H(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} h[n] \exp(-j\omega n) \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |h[n]| \, |\exp(-j\omega n)|$$

$$= \sum_{n=-\infty}^{\infty} |h[n]| = \| h[n] \|_1$$

Consequently:

$$\underline{\| h[n] \|_1 \geq \| H(e^{j\omega}) \|_\infty \qquad (c)}$$

Combining (a), (b), & (c):

$$\| h[n] \|_2 \leq \| H(e^{j\omega}) \|_\infty \leq \| h[n] \|_1$$

or

$$\frac{1}{\| h[n] \|_2} \geq \frac{1}{\| H(e^{j\omega}) \|_\infty} \geq \frac{1}{\| h[n] \|_1}$$