6 Impulse Sampling and Nyquist Sampling Theorem

To facilitate the processing of a continuous-time signal $x_c(t)$, it is more convenient to work with discretised samples taken at integer multiples of the sampling period $t = nT_s$. The discrete sequence so obtained by sampling the continuous-time signal $x_c(t)$ is denoted $x[n]$ so that $x[n] = x_c(nT_s)$.

The process of sampling the continuous-time signal $x_c(t)$ is modeled as multiplication of the the continuous-time waveform with a periodic impulse train:

$$p(t) \quad = \quad \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

whose period is T_s:

$$x_s(t) \quad = \quad x_c(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \quad = \quad \sum_{k=-\infty}^{\infty} x_c(kT_s)\delta(t - kT_s) \;=\; \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT_s).$$

Our goal is to determine the relation of sampled signal spectrum to the spectrum of the continuous-time signal. The first approach is to use the product-convolution Fourier transform pair:

$$X_s(\Omega) \;=\; \frac{1}{2\pi} \left\{ X_c(\Omega) * P(\Omega) \right\}.$$

The periodic impulse train $p(t)$ has a Fourier series expansion and its Fourier coefficient is $c_k = \frac{1}{T_s}, \forall k$. Using the expression for the Fourier transform of a periodic signal we obtain the transform of the impulse train as:

$$P(\Omega) \;=\; \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T_s}\right).$$

The spectrum of the sampled signal can therefore be written as:

$$\begin{aligned} X_s(\Omega) \quad &= \quad \frac{1}{2\pi} \left\{ X_c(\Omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T_s}\right) \right\} \\ X_s(\Omega) \quad &= \quad \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T_s}\right) \;=\; \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\Omega_s\right). \end{aligned} \tag{1}$$

Equation 1 is significant in that it tells us that the spectrum of the sampled signal is a sum of replicas of itself shifted in frequency by $\Omega_s = \frac{2\pi}{T_s}$. This relation is often refered to as the *spectral aliasing sum*. Constructive and destructive interference of the terms in the sum is refered to as *spectral aliasing*. If each term in the aliasing sum is not restricted in terms of its support in the frequency domain then the terms in the aliasing sum will overlap and cause loss of information via constructive and destructive interference. So impulse sampling of the continuous-time signal $x_c(t)$ will result in loss of information unless the terms in the aliasing sum are restricted in terms of their frequency-domain support, i.e., they must be band-limited.

Let us assume that $x_c(t)$ is indeed a band-limited continuous-time signal band-limited to $-\Omega_m \leq \Omega \leq \Omega_m$. The copy of $X_c(\Omega)$ shifted by Ω_s, i.e., $X_c(\Omega - \Omega_s)$, has a freqyency-domain support between $\Omega_s - \Omega_m \leq \Omega \leq \Omega_s + \Omega_m$. There will be no overlap between $X_c(\Omega - \Omega_s)$

and $X_c(\Omega)$ if the lower band edge of $X_c(\Omega - \Omega_s)$ does not fall within the band $0 \leq \Omega \leq \Omega_m$, i.e., for no aliasing and loss of information we require that:

$$\Omega_s - \Omega_m \geq \Omega_m \longrightarrow \Omega_s \geq 2\Omega_m.$$

The lowerbound $\Omega_s = 2\Omega_m$ is commonly refered to as the the *Nyquist sampling frequency*. This means that the minimum angular sampling frequency Ω_s needed for no information loss is the Nyquist rate, i.e., twice the maximum frequency content of the continuous-time signal $x_c(t)$. Reformulated this means for a given sampling period T_s, the maximum allowable frequency content of the continuous-time signal $x_c(t)$, Ω_m is :

$$\Omega_m \quad \leq \quad \frac{\pi}{T_s}.$$

The other approach to obtaining the spectrum of the sampled signal $X_s(\Omega)$ is to use the discrete-sequence $x[n]$:

$$\begin{aligned} X_s(\Omega) &= \mathbf{F}\left(\sum_{k=-\infty}^{\infty} x[k]\delta(t-kT_s)\right) = \sum_{k=-\infty}^{\infty} x[k]\mathbf{F}(\delta(t-kT_s)) \\ X_s(\Omega) &= \sum_{k=-\infty}^{\infty} x[k]\exp\left(-j\Omega kT_s\right) = X(e^{j\omega}), \text{ where } \quad \omega = \Omega T_s. \end{aligned} \tag{2}$$

The spectrum of the sampled signal using the sequence $x[n]$ is periodic in the Ω variable with fundamental period Ω_s as described by:

$$\begin{aligned} X_s(\Omega + \Omega_s) &= \sum_{k=-\infty}^{\infty} x[k]\exp\left(-j\left(\Omega + \Omega_s\right)kT_s\right) \\ X_s(\Omega + \Omega_s) &= \sum_{k=-\infty}^{\infty} x[k]\exp\left(-j\Omega kT_s\right)\underbrace{\exp\left(-j2k\pi\right)}_{1} = X_s(\Omega). \end{aligned}$$

Since the function $X_s(\Omega)$ described above is periodic in the frequency variable Ω with period Ω_s, it has a complex Fourier series representation in the frequency-domain as:

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-jn\left(\frac{2\pi}{\Omega_s}\right)\Omega\right) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-jn\Omega T_s\right),$$

where the Fourier coefficients in the expansion are obtained from:

$$c_k = \frac{1}{\Omega_s}\int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} X_s(\Omega)\exp\left(jk\Omega T_s\right)d\Omega.$$

Substituting Eq. 2 into the Fourier coefficient expression and interchanging the order of the sum and integral we have:

$$c_k = \frac{1}{\Omega_s}\sum_{n=-\infty}^{\infty} x[n]\underbrace{\int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}}\exp\left(j(k-n)\Omega T_s\right)d\Omega}_{\Omega_s\delta_{n,k}} = \frac{1}{\Omega_s}\sum_{k=-\infty}^{\infty} x[n]\left(\Omega_s\delta_{n,k}\right) = x[k].$$

This means that the sequence of Fourier coefficients of the periodic function $X_s(\Omega)$ is just the discrete sequence $x[k]$ of sampled values of the continuous-time signal $x_c(t)$. In other words the DTFT relation in Eq. 2 is also the complex Fourier series representation of $X_s(\Omega)$.

7 Signal Recovery from Samples

The Nyquist sampling theorem developed in the previous section says that there is no loss of information in the impulse sampling process if we bandlimited the signal to the band: $-\frac{\pi}{T_s} \leq \Omega \leq \frac{\pi}{T_s}$ and we then sample this bandlimited signal at the Nyquist rate or higher.

Our objective in this section is to recover the continuous-time signal $x_c(t)$ from the discrete-time samples $x[k]$. We will do that by noting that when there is no loss of information we can lowpass filter the sampled signal $x_s(t)$ to remove the extra spectral copies of $X_c(\Omega)$ at the frequencies: $\Omega = k\Omega_s, k \neq 0$ as suggested by:

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\Omega_s\right).$$

This lowpass filter $H_i(\Omega)$ is defined by:

$$H_i(\Omega) = \begin{cases} T_s & -\frac{\pi}{T_s} \leq \Omega \leq \frac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

The impulse response of the lowpass filter, denoted as $h_i(t)$, found through inverse Fourier transformation is:

$$h_i(t) = \text{Sa}\left(\frac{\pi t}{T_s}\right) = \text{Sinc}\left(\frac{t}{T_s}\right).$$

The output of the lowpass filter denoted as $\hat{x}_c(t)$ can be written as:

$$\hat{x}_c(t) = x_s(t) * h(t) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT_s) * h_i(t) = \sum_{k=-\infty}^{\infty} x[k]h_i(t - kT_s).$$

If the signal $x_c(t)$ was bandlimited and was sampled at a rate greater than the Nyquist rate then the signal $x_c(t)$ can be recovered as:

$$x_c(t) = \sum_{k=-\infty}^{\infty} x[k] \text{ Sinc}\left(\frac{t}{T_s} - k\right).$$

The sinc function in the above expression can be thought of as an interpolating function that interpolates between the discrete samples. The process of converting $x[k]$ into $x_c(t)$ using the lowpass filter is often refered to as *sinc interpolation*. Specifically the sinc functions in the interpolation formula are true interpolation functions because

$$x_c(mT_s) = \sum_{k=-\infty}^{\infty} x[k] \text{ Sinc}\left(\frac{mT_s}{T_s} - k\right) = x[m].$$

The fact that the interpolation formula gives correct results for other time instants also can be infered from the frequency domain argument from the previous section.

If the signal $x_c(t)$ is not completely bandlimited to the desired region then we need to lowpass filter the signal with an *anti-alias lowpass filter*, $H_a(\Omega)$, that restricts its spectral content to the desired band if we are to recover the continuous-time signal from its samples.

8 Alternative View of Sampling Theorem

If the continuous time signal $x_c(t)$ has finite energy, i.e.,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \Longleftrightarrow x(t) \in \mathbf{H}([a,b]).$$

then via Parsevals theorem we can see that its Fourier transform $X_c(\Omega)$ is also a finite energy signal, i.e.,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega < \infty \Longleftrightarrow X(\Omega) \in \mathbf{H}([-\infty,\infty]).$$

If in addition we impose the restriction that $X_c(\Omega)$ is bandlimited to $|\Omega| \leq \frac{\pi}{T_s}$ then

$$X_c(\Omega) \in \mathbf{H}\left(\left[-\frac{\pi}{T_s}, \frac{\pi}{T_s}\right]\right).$$

The Nyquist sampling theorem of the previous section can then be reformulated as a transformation from the Hilbert space of finite–energy, bandlimited signals, $\mathbf{H}([-\frac{\pi}{T_s}, \frac{\pi}{T_s}])$ to the space of square summable sequences $x[n] \in l_2$ via the reconstruction relation:

$$x_c(t) \quad = \sum_{k=-\infty}^{\infty} x[k] \ \mathrm{Sinc}\left(\frac{t}{T_s} - k\right).$$

Specifically if we define a basic or *mother* function $\phi(t)$ as

$$\phi(t) = \mathrm{Sinc}\left(\frac{t}{T_s}\right)$$

then the other functions $\phi_k(t)$ in the summation can be obtained as shifts of this basic function via :

$$\phi_k(t) = \phi(t - kT_s).$$

Furthermore note that these functions $\phi_k(t)$ are orthogonal because of Parsevals theorem:

$$< \phi_m(t), \phi_n(t) > = \frac{T_s^2}{2\pi} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} \exp\left(-j\Omega(m-n)T_s\right) d\Omega = T_s \delta_{mn}.$$

The sinc functions in the summation part of the reconstruction formula

$$\phi_k(t) = \mathrm{Sinc}\left(\frac{t}{T_s} - k\right)$$

constitute an orthogonal basis for the space of signals whose Fourier transform is finite–energy and bandlimited, i.e., $X_c(\Omega) \in \mathbf{H}([-\frac{\pi}{T_s}, \frac{\pi}{T_s}])$.

From the reconstruction formula it is easy to show that if $x_c(t)$ is a finite energy bandlimited signal then the sequence $x[n]$ is a square summable sequence, i.e., $x[n] \in l_2$ because:

$$E_x = ||x_c(t)||^2 = T_s^2 \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$