ECE-539, Spring 2005
Digital Signal Processing

Sensitivity to Quantization

Look at Direct Form:

Define

$$B(z) \triangleq \sum_{k=0}^{M} b[k]\, z^{-k}$$

$$= b_0 \prod_{k=1}^{M} (1 - z_k z^{-1})$$

$$A(z) \triangleq 1 - \sum_{k=1}^{N} a[k]\, z^{-k}$$

$$= \prod_{k=1}^{N} (1 - p_k z^{-1})$$

$$\frac{\partial A(z)}{\partial z} = \sum_{r=1}^{N} \prod_{\substack{k=1 \\ k \neq r}}^{N} \frac{p_k}{z^2} (1 - p_k z^{-1})$$

$$\left.\frac{\partial A(z)}{\partial z}\right|_{z = p_i} = \frac{p_i}{p_i^2} \prod_{\substack{k=1 \\ k \neq i}}^{N} (1 - p_k p_i^{-1})$$

$$= \frac{1}{p_i^N} \prod_{\substack{k=1 \\ k \neq i}}^{N} p_i - p_k \quad , i = 1 \ldots N$$

Similarly

$$\left.\frac{\partial B(z)}{\partial z}\right|_{z = z_j} = \frac{b_0}{z_j^N} \prod_{\substack{k=1 \\ k \neq j}}^{N} z_j - z_k$$

$$i = 1, 2, \ldots M$$

$$\frac{\partial A(z)}{\partial a[k]} = \frac{\partial}{\partial a[k]} \left\{ 1 - \sum_{p=1}^{N} a[p] z^{-p} \right\}$$

$$= -z^{-k}, \quad k = 1, 2, 3, \ldots\ldots N$$

Similarly

$$\frac{\partial B(z)}{\partial a[k]} = z^{-k}, \quad k = 1, 2, \ldots\ldots M$$

$$\left. \frac{\partial A(z)}{\partial a[k]} \right|_{z = p_i} = -p_i^{-k}$$

$$\left. \frac{\partial B(z)}{\partial b[k]} \right|_{z = z_j} = +(z_j)^{-k}$$

Using Chain rule:

$$\frac{\partial p_i}{\partial a[k]} = \left. \frac{\partial A(z)}{\partial a[k]} \right|_{z = p_i} \Bigg/ \left. \frac{\partial A(z)}{\partial z} \right|_{z = p_i}$$

$$= -p_i^{-k} \Bigg/ \frac{1}{p_i^{N}} \prod_{\substack{r=1 \\ r \neq i}}^{N} p_i - p_r$$

$$\frac{\partial p_i}{\partial a[k]} = -p_i^{N-k} \Bigg/ \prod_{\substack{r=1 \\ r \neq i}}^{N} p_i - p_r$$

$$1 \leq i \leq N$$

$$1 \leq k \leq N$$

Similarly

$$\frac{\partial z_j}{\partial b[k]} = + \frac{(z_j)^{N-k}}{b_0} \frac{1}{\prod_{\substack{r=1 \\ r \neq j}}^{N} (z_j - z_r)}$$

Observations

- $$\frac{\partial p_i}{\partial a[k]} \propto \frac{1}{p_i - p_r}, \quad \begin{matrix} r = 1, \ldots N \\ r \neq i \end{matrix}$$

  $$\frac{\partial z_j}{\partial b[k]} \propto \frac{1}{z_j - z_r}, \quad \begin{matrix} r = 1, 2, \ldots M \\ r \neq j \end{matrix}$$

$\Rightarrow$ Clustering of poles and zeroes produces larger coefficient sensitivities

- $$dp_i = \sum_{k=1}^{N} \frac{\partial p_i}{\partial a[k]} \, da[k]$$

  $$dz_j = \sum_{k=1}^{M} \frac{\partial z_j}{\partial b[k]} \, db[k]$$

* Quantization of one coefficient produces error in all pole-zero locations

* Error in pole-zero locations increases with # of quantized coefficients