Solution to PS #1 , Spring 2004
Digital Signal Processing, EECE-539
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This problem looks at a simple proof of the uncertainty principle for a specific class of signals. The first step in solving the problem is to use the Cauchy-Schwartz inequality for two functions $f(t)$ and $g(t)$ that belong to the Hilbert space of square integrable signals:

$$\left| \int_{-\infty}^{\infty} f(t) g^*(t) dt \right|^2 \leq \int_{-\infty}^{\infty} |f(t)|^2 dt \int_{-\infty}^{\infty} |g(t)|^2 dt.$$

specifically when $f(t) = tx(t)$ and $g(t) = \frac{dx}{dt}$ this reduces to :

$$\left| \int_{-\infty}^{\infty} tx(t) \frac{dx}{dt} dt \right|^2 \leq \int_{-\infty}^{\infty} |tx(t)|^2 dt \int_{-\infty}^{\infty} \left| \frac{dx}{dt} \right|^2 dt. \quad (1)$$

The second fact to note is Parsevals theorem that relates the norm of the signal $x(t)$ in the time-domain to its norm in the frequency domain. Specifically:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \longleftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega.$$

Upon applying Parsevals to the derivative of the signal we obtain:

$$\int_{-\infty}^{\infty} \left| \frac{dx}{dt} \right|^2 dt \longleftrightarrow \int_{-\infty}^{\infty} \Omega^2 |X(\Omega)|^2 d\Omega. \quad (2)$$

The frequency dispersion of the signal $x(t)$ can now expressed in terms of time-domain quantities as:

$$D_\Omega = \frac{\int_{-\infty}^{\infty} \Omega^2 |X(\Omega)|^2 d\Omega}{\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega} = \frac{\int_{-\infty}^{\infty} \left| \frac{dx}{dt} \right|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}. \quad (3)$$

We are now ready to look at the product of the time and frequency dispersion of the signal $x(t)$:

$$D_t D_\Omega = \frac{\int_{-\infty}^{\infty} \left| \frac{dx}{dt} \right|^2 dt \int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\left(\int_{-\infty}^{\infty} |x(t)|^2 dt \right)^2} \geq \frac{\left| \int_{-\infty}^{\infty} tx(t) \frac{dx}{dt} dt \right|^2}{\left(\int_{-\infty}^{\infty} |x(t)|^2 dt \right)^2} \quad (4)$$

The numerator of the RHS of the inequality above can be evaluated via integration by parts as:

$$\int_{-\infty}^{\infty} tx(t)\frac{dx}{dt}dt = \left[tx^2(t)/2\right]_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} |x(t)|^2 dt$$

We now incorporate the information about $x(t)$, i.e., $x(t) \approx o(t^{-3})$ into the above to get:

$$\int_{-\infty}^{\infty} tx(t)\frac{dx}{dt}dt = -\frac{1}{2}||x(t)||^2 \tag{5}$$

Incorporating this result into the RHS of the inequality we get :

$$D_t D_\Omega \geq \frac{1}{4} \tag{6}$$

The equality portion of the CS inequality holds good when $f(t) = Kg(t),\ K \in \mathbf{R}, K > 0$. The signals for which this is the case are given by:

$$tx(t) = -K\frac{dx}{dt},$$

where we have used the information that $x(t)$ decays rapidly to zero into the negative sign. The solution for the equality portion yields the signal:

$$x(t) = ce^{-Kt^2/2}$$

Normalizing $x(t)$ so that it has unit norm yields the expression:

$$x(t) = \sqrt{\frac{K}{2\pi}}e^{-Kt^2/2} \tag{7}$$