Quantization Via the CDF Method

Consider a source signal $x[n]$ that is exponentially distributed with mean $\frac{1}{\lambda}$ and a PDF of the form:

$$f_X(x) = \lambda \exp(-\lambda x) U(x)$$

Using the CDF transformation discussed in the class we can transform the exponentially distributed source into a uniformly distributed source using the transform:

$$z[n] = (1 - \exp(-\lambda x[n])) U(x[n])$$

so that $z[n] \sim U([0, 1])$. If we now quantize the signal $z[n]$ using a uniform quantization scheme we obtain a signal $z_q[n]$ such that:

$$z_q[n] = z[n] + e_q[n], \quad e_q[n] \sim U\left(\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]\right).$$

Since $e_q[n]$ is uniformly distributed, the quantization error variance is given by:

$$\sigma_e^2 = \frac{\Delta^2}{12} = \frac{X_{\max}^2}{2^{2B}12} = \frac{1}{2^{2B}12}.$$

After transforming the quantized signal $z_q[n]$ using the inverse CDF function the output signal is given by:

$$\begin{aligned} x_q[n] &= \frac{\log(1 - z[n] - e_q[n])}{-\lambda} = \frac{\log\left((1 - z[n])\left(1 - \frac{e_q[n]}{1 - z[n]}\right)\right)}{-\lambda} \\ x_q[n] &= \frac{\log((1 - z[n]))}{-\lambda} - \frac{\log\left(1 - \frac{e_q[n]}{1 - z[n]}\right)}{\lambda} \end{aligned}$$

Using the fact, that for a continuous RV, the CDF is a bijective map, we can write:

$$x_q[n] = F_X^{-1}(F_X(x[n])) + \frac{\log\left(1 - \frac{e_q[n]}{1 - z[n]}\right)}{-\lambda} = x[n] + \frac{\log\left(1 - \frac{e_q[n]}{1 - z[n]}\right)}{-\lambda}, \quad |z[n]| < 1.$$

The output quantization error sequence in this case is given by:

$$e_{\text{out}}[n] = \frac{\log\left(1 - \frac{e_q[n]}{1 - z[n]}\right)}{-\lambda}, \quad |z[n]| < 1$$

Specifically when the number of bits in the representation of $z[n]$ is sufficiently large then using the expansion for $\log(1 - x), \quad |x| < 1$:

$$e_{\text{out}}[n] \approx \frac{e_q[n]}{1 - z[n]}, \quad |z[n]| < 1$$

Since the signal $z[n]$ is uniformly distributed (a continuous RV), the probability of getting a value of $|z[n]| = 1$ is zero. We can further avoid this situation by using a saturation type uniform quantizer that rounds down. Note that this method can be applied to any source distribution that has a continuous PDF. The quantization scheme however, does not incorporate the fact that the source distribution is exponential. The quantization scheme is therefore suboptimal in the sense that each of the quantizer intervals and quantization levels is given equal importance in terms of bit-loading eventhough they are not equally likely. The Lloyd-Max quantizer on the other hand is optimal in the sense that it produces the quantizer parameters that are optimal in the MMSE sense for a source signal with a specific distribution.

```
%*************************************************
% MATLAB file for quantizing an input source which
% is exponential distribution with
% parameter lamda
% AUTHOR: Balu Santhanam
% DATE  : 10/25/01
%*************************************************
%  Source symbols are exponentially distributed
lamda = 0.5; x = rndv(1000,lamda,'expn');
y = 1 - exp(-lamda*x);
% y has a uniform distribution
% quantize it to B bits
B_var = [2:2:16];
for i = 1:1:length(B_var)
    [y_q,alphab,SN] = fxquant(y,0,1,B_var(i),3);
    e_q = y - y_q;
% Transform the uniform source back to a exponential
% source
    x_out = log(1 - y_q)/(-lamda);
    alpha_out = log(1 - alphab)/(-lamda);
    e_out = x - x_out;
    clear alpha_out e_q y_q
    SNR(i) = 20*log10(std(x_out)/std(e_out));
end
```

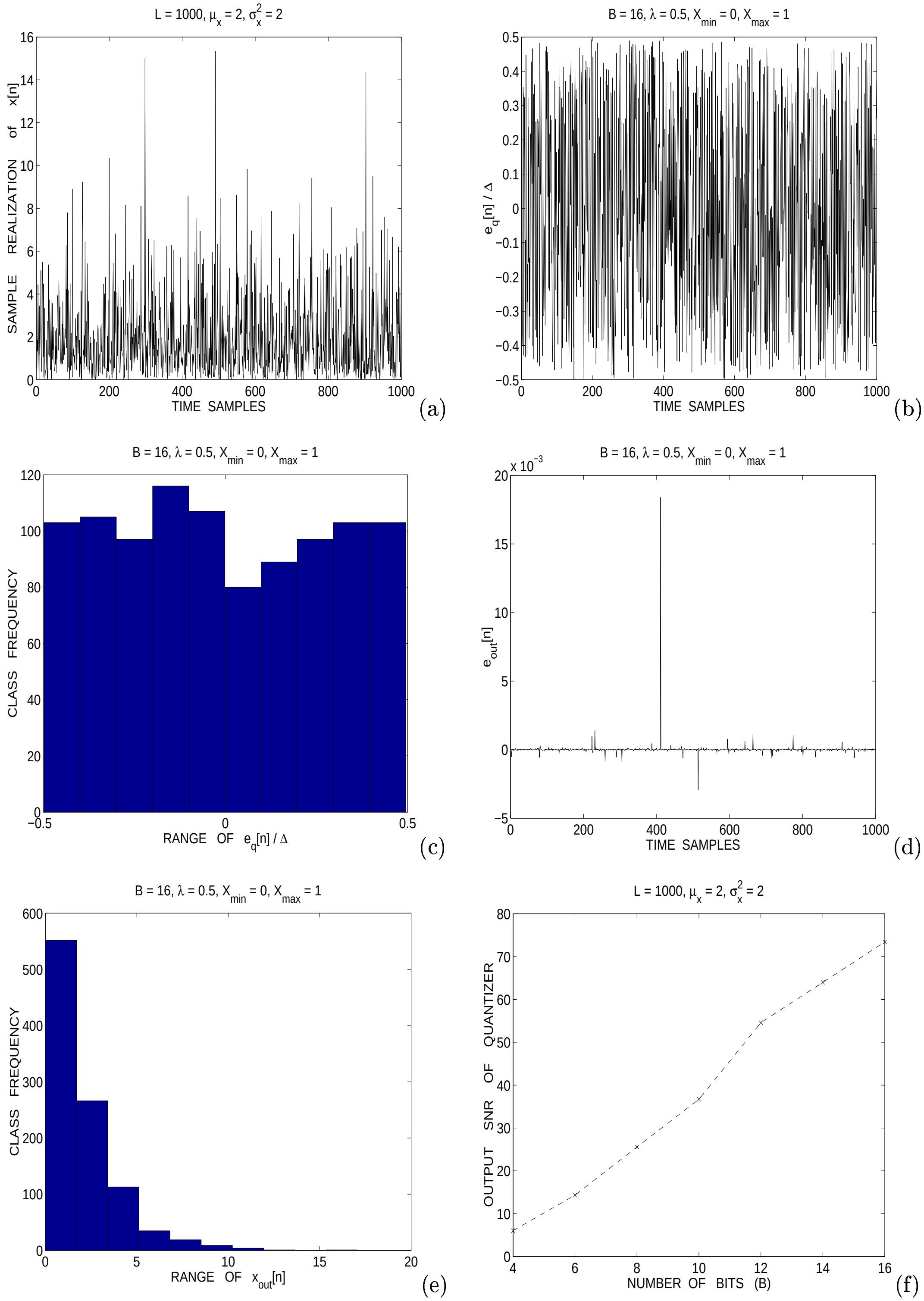

Figure 1: Sample output: (a) sample realization of the source signal $x[n]$, (b) normalized uniform quantization error signal, (c) histogram of the uniform quantization error signal, (d) output quantization error $e_{\text{out}}[n]$, (e) histogram of quantized output $x_{\text{out}}[n]$, SNR of the quantizer for different number of bits in the output alphabet.