ECE - 539, SPRING 2014
DIGITAL SIGNAL PROCESSING

EXAMPLE: Zero padding & Resolution

Consider a discrete-time sequence $x[n]$ given by:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

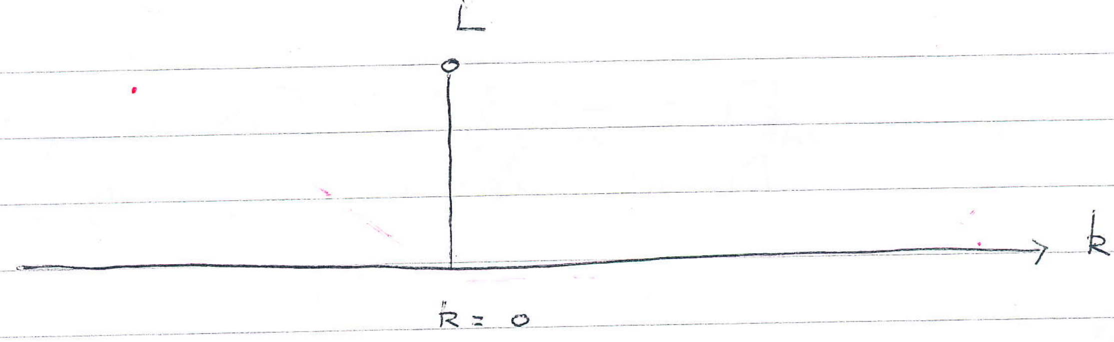
Assuming for now that $L \neq 2^{\gamma}$ and $L < N$:

$$X[k] = \sum_{n=0}^{L-1} x[n] W_L^{nk}, \quad 0 \leq k \leq N-1$$

Substituting $x[n]$ into the expression above:

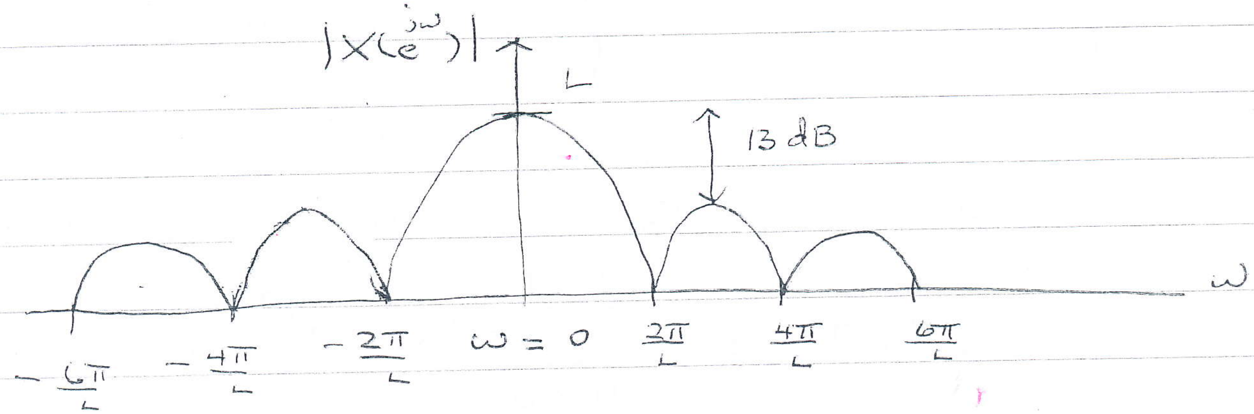
$$X[k] = \sum_{n=0}^{L-1} 1 W_L^{nk} = \frac{1 - W_L^{kL}}{1 - W_L^{k}}$$

$$= L\,\delta[((k))_L] = L\,\delta[k + rL], \quad r \in \pm I$$



The corresponding DTFT:

$$X(e^{j\omega}) = e^{-j\omega\left(\frac{L-1}{2}\right)} \frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$



The DFT in this case corresponds to a valid sample of the DTFT at $\omega = 0$

Now consider the sequence which is zero-padded to $N = 2^8$

$$Y[k] = \sum_{n=0}^{L-1} x[n] W_N^{nk} = \frac{1 - W_N^{Lk}}{1 - W_N^{k}}$$

$$= e^{-j\frac{\pi k}{N}\left(\frac{L-1}{2}\right)} \sin\left(\frac{\pi L k}{N}\right) \Big/ \sin\left(\frac{\pi k}{N}\right)$$

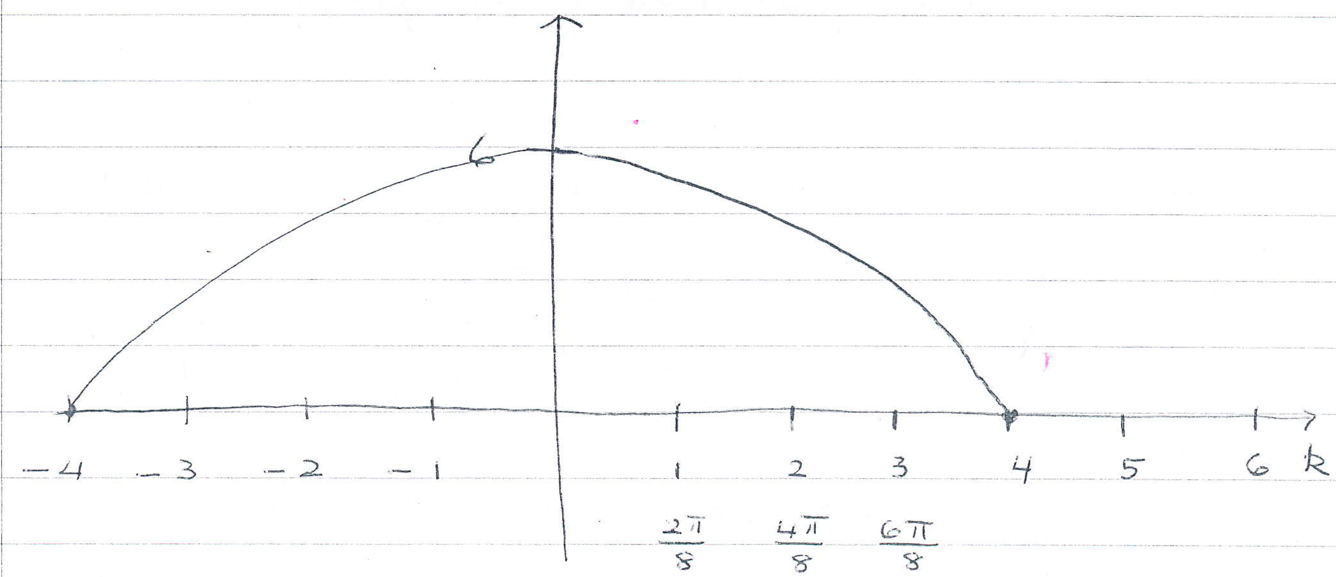
For simplicity sake :

$L = 6$

$N = 8$

$$Y[k] = e^{-j\frac{\pi k}{8}\frac{5}{2}} \frac{\sin\left(\frac{\pi 6 k}{8}\right)}{\sin\left(\frac{\pi k}{8}\right)}$$

$$= e^{-j\frac{\pi k}{16}\cdot 5} \frac{\sin\left(\frac{3 k \pi}{4}\right)}{\sin\left(\frac{\pi k}{8}\right)}$$



The same valid sample of $X(e^{j\omega})$

→ appears at $k = 0$ or $w = 0$

→ Periodicity of spectrum for $N = 8$
with peak value 6 at $k = 8, 16, 24$ etc

Zero-padding provides the
→ appearance that there are more
direct samples

→ The extra samples corresponds to
interpolation between direct samples

→ $\omega_k = \frac{2\pi}{L} k, \quad 0 \le k \le L-1$

are the direct samples.