Spectral Zoom Operation

In our previous discussion we have mentioned that the decimation operation corresponds to a spectral zoom operation. In this section we will analyze this operation in detail. The decimation operation by a factor M as seen before is equivalent to the cascade of a ideal lowpass filter $H_{\text{lp}}(z)$, with unity passband gain and a cut-off frequency $\omega_c = \frac{\pi}{M}$ with a downsampling operation by a factor of M. The time-domain input-output characteristic of this operation is given by:

$$y_d[n] = \sum_{k=-\infty}^{\infty} x[k] h_{\text{lp}}[Mn - k]$$

In the Fourier or DTFT domain this relation becomes:

$$Y_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2\pi i}{M}\right)}\right) H_{\text{lp}}\left(e^{j\left(\frac{\omega - 2\pi i}{M}\right)}\right).$$

Although the above relation contains filtered version of aliased spectral copies of the input $x[n]$, the zoom operation is still not explicit. Instead let us look at each operation a little closer in the frequency domain.

Zooming

The lowpass filtering operation by definition removes the high frequency content of the input signal and retains just the low frequency components. During this operation the bandwidth of the signal $x[n]$ has been reduced by a factor of M and corresponds to a *spectral zooming* operation into the low frequency spectral band $\omega \in \left[0, \frac{\pi}{M}\right]$ as given by:

$$X_{\text{lp}}(e^{j\omega}) = X(e^{j\omega}) H_{\text{lp}}(e^{j\omega}).$$

Note that the filtered signal $x_{\text{lp}}[n]$ is not a full-band signal, i,e, has non-zero spectral content only over a subset of $\omega \in [-\pi, \pi]$ and will be refered to as a *subband* signal.

Magnification

The operation of downsampling as seen before aliases the spectrum of $x_{\text{lp}}[n]$ according to:

$$Y_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X_{\text{lp}}\left(e^{j\left(\frac{\omega - 2\pi i}{M}\right)}\right).$$

Note that each of the spectral copies in the sum has been frequency-scaled by M and consequently has been scaled by M. Each of these terms in the sum is therefore periodic with a frequency-based periodicity of $2M\pi$ and have a spectral amplitude of $\frac{1}{M}$. These copies when superimposed will not overlap because of the fact that $x_{\text{lp}}[n]$ is not a full-band signal. Hence the resultant spectrum over the interval $\omega \in [-\pi, \pi]$ will just be the $k = 0$ term in the sum. On closer observation we can see that the $k = 0$ term in the sum, i.e.,

$$Y_d(e^{j\omega}),\ \omega \in [-\pi, \pi] = Y_d^{(0)}(e^{j\omega}) = \frac{1}{M} X_{\text{lp}}(e^{j\frac{\omega}{M}}).$$

is exactly the spectrum of the lowpass filtered signal $x_{\text{lp}}[n]$ that has been magnified or blown-up spectrally to fit the region $\omega \in [-\pi, \pi]$. In other words, the partial-band signal $x_{\text{lp}}[n]$ is being magnified to a full-band signal. So the second operation of downsampling the lowpass filtered signal can be interpreted as a *spectral magnification* operation.

The filter used in the operation does not need to be a lowpass filter. If the lowpass filter were replaced by a bandpass filter or a highpass filter the resultant operation would correspond to a spectral zoom and magnification into the corresponding bandpass/highpass subbands of the signal.