EECE - 539, Digital Signal Processing

Spring 2001, Notes on Review

Z - transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Region of Convergence:

$$ROC = \{ z \in \mathbb{C} : |X(z)| < \infty \}$$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n}$$

If $\sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty$

then $|X(z)| < \infty$ (No poles in R.O.C.)

Convergence of $X(z)$ depends only on $|z|$ and not on $\measuredangle z$ (Annular rings)

A Z-transform expression must be accompanied by a R.O.C.

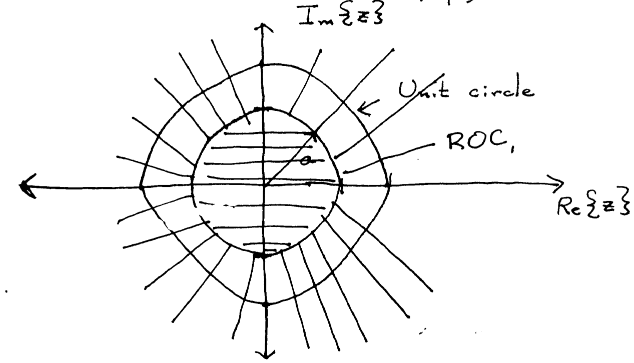
Reason: $x[n] = a^n u[n]$, $0 < |a| < 1$

$$X(z) = \frac{1}{1 - a z^{-1}}, \quad |z| > |a| \quad (ROC_1)$$

$$x[n] = -a^n u[-n-1]$$

$$X(z) = \frac{1}{1 - a z^{-1}}, \quad |z| < |a| \quad (ROC_2)$$

$|a| > 1$



Both sequences have identical $X(z)$

- Only differ by R.O.C.

- R.O.C. specification makes the Z-transformation unique

Unit - Circle :

$$X\left(z = e^{j\omega},\ |\omega| \le \pi\right)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(e^{j\omega})$$

$\Rightarrow$ DTFT is just the Z-transform evaluated along $|z| = 1$ or UC

$\Rightarrow$ For $x[n]$ to be an absolutely summable, $X(e^{j\omega})$ should exist

$\Rightarrow$ U.C. $\subset$ ROC $\Rightarrow$ $x[n]$ is a stable sequence

Causality :

If $x[n]$ is a causal sequence

$x[n] = 0, \quad n < 0$

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$\Rightarrow$ $X(z)$ contains only $z^{0}, z^{-1}, \ldots\ z^{-n}, \ldots\ldots$ or no positive powers of z

$\Rightarrow$ $\lim\limits_{z\to\infty} X(z) = \lim\limits_{z\to\infty} x[0] + x[1]z^{-1} + \ldots\ldots$

$= x[0]$

$\Rightarrow$ Anticausal sequences will have Z-transforms with no $-$ve powers of z

$\Rightarrow$ If $x[n]$ is a finite length sequence then a causal sequence $x[n]$ will have a pole at $z = 0$

i.e., $\lim\limits_{z\to 0} \sum\limits_{n=0}^{L-1} x[0]\, z^{-n} = \infty$

$\Rightarrow$ Causal Sequences have a doenut shaped R.O.C.

e.g. $x[n] = a^{n} u[n]$

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

Anticausal sequences have a disk shaped R.O.C.

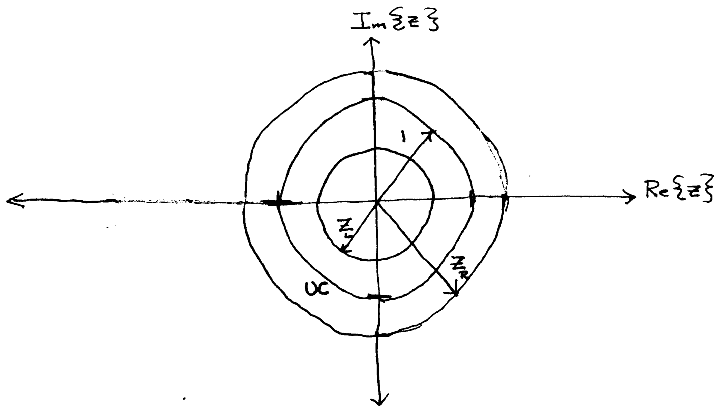
e.g. $x[n] = -a^{n} u[-n-1]$

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$\Rightarrow$ A stable & causal sequence $x[n]$ has a $X(z)$ will all poles within U.C.

If $x[n]$ is a dual-sided sequence then

- $x[n]$ has support for both +ve and −ve indices "n".
- $X(z)$ contains both power of z^{-1}, and powers of z
- ROC is in the form of an annular ring, i.e, $z_L \le |z| \le z_R$



- Finite length dual-sided sequences $x[n]$ will have a pole at $z = 0$ & a pole at $z = \infty$.

Properties of the Z-transform

(a) $$Z\{x[n-n_0]\} = z^{-n_0} X(z)$$

$$\{ROC\}_{new} = \begin{cases} \{ROC\}_{old} \setminus z=0, & n_0>0 \\ \{ROC\}_{old} \setminus z=\infty, & n_0<0 \\ \{ROC\}_{old}, & n_0=0 \end{cases}$$

(b) $$Z\{x[-n]\} = X(1/z)$$

$$\{ROC\}_{new} = \frac{1}{\{ROC\}_{old}}$$ (Reciprocal Locations)

(c) $$Z\{x^*[-n]\} = X^*(1/z)$$

$$\{ROC\}_{new} = \frac{1}{\{ROC\}^*_{old}}$$ (Conjugate Reciprocal)

(d) $$Z\{z_0^n x[n]\} = X\left(\frac{z}{z_0}\right)$$

$$\{ROC\}_{new} = z_0 \{ROC\}$$

(e) $$X(z=1) = \sum_{n=-\infty}^{\infty} x[n] = X(e^{j0})$$

D.C. value theorem

(f) $$Z\{x_1[n] * x_2[n]\} = X_1(z) X_2(z)$$

$ROC : ROC^{6} \cap ROC$