LTI Systems and Random Signals

Consider a LTI system with a transfer function $H(s)$ which is excited with a WSS random signal $X(t)$, with mean μ_x, variance σ_x^2, and ensemble ACF $R_{xx}(\tau)$. We will further assume that average power of the random signal $P_{\text{ave}} = \sigma_x^2 < \infty$ and that the random signal $X(t)$ is continuous in the MS sense, i.e.,

$$\lim_{t \to t_o} E\{(X(t) - X(t_o))^2\} = 0.$$

This in essence means that the expectation operator and the limit operator and hence an integral operator can be interchanged.

Output Random Signal

Our goal in this section is to study the statistical characerestics of the output of the LTI system $Y(t)$. We begin by writing the ouput of the system via the convolution theorem:

$$Y(t) = \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau. \tag{1}$$

This relation can again be interpreted in two ways:

1. the i^{th} sample function of the output random process $Y(t)$ is obtained by the convolution of the i^{th} sample function of the input random process $X(t)$ with the impulse response of the LTI system $h(\tau)$.
2. the constituent random variables $Y(t), t \in \mathbf{R}$ of the output are obtained as a linear combination of the constituent random variables $X(t), t \in \mathbf{R}$ of the input.

Henceforth, when we refer to the processing of a random signal via a LTI system or the filtering of a random signal we will refer to either of these viewpoints.

Output Ensemble Statistics

The mean of the output random signal in either case is given by:

$$E(Y(t)) = E\left\{\int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau\right\} = \mu_x \int_{-\infty}^{\infty} h(\tau) d\tau = \mu_x H(0). \tag{2}$$

This relation is significant in terms of its implication that the mean of the ouput process is also a constant. For simplicity sake in our analysis let us further assume that the input signal has zero mean. The cross-correlation function between the output $Y(t)$ and the input $X(t)$ is given by:

$$\begin{aligned} R_{yx}(\tau) &= E\left\{\int_{-\infty}^{\infty} h(\sigma) X(t-\sigma) d\sigma X^*(t-\tau)\right\} \\ R_{yx}(\tau) &= \int_{-\infty}^{\infty} h(\sigma) R_{xx}(\tau-\sigma) d\sigma = R_{xx}(\tau) * h(\tau). \end{aligned} \tag{3}$$

The ACF of the output random process $Y(t)$ is obtained via:

$$\begin{aligned} R_{yy}(\tau) &= E\left\{Y(t) \int_{-\infty}^{\infty} h^*(\rho) X^*(t-\tau-\rho) d\rho\right\} \\ R_{yy}(\tau) &= \int_{-\infty}^{\infty} h^*(\rho) R_{yx}(\tau+\rho) d\rho = R_{yx}(\tau) * h^*(-\tau). \end{aligned} \tag{4}$$

Relating the ACF of the output process $Y(t)$, $R_{yy}(\tau)$ and the ACF of the input, $R_{xx}(\tau)$ we have:

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h^*(-\tau). \tag{5}$$

These relations are significant because they imply that the ACF of the output random signal $Y(t)$ is dependent only on the difference between the sampling instants, i.e., the output random signal $Y(t)$ is also WSS. Note that this relation has a double convolution operation. Analytical evaluation of the double convolution becomes cumbersome even in the simple cases.

Output Power Spectra

Instead we seek to circumvent the convolutions by transforming the equations obtained to the frequency domain. Taking the bilateral Laplace transform on both sides of Eq. (3) we have:

$$\boxed{P_{yx}(s) = P_{xx}(s)H(s), \quad s \in \mathbf{ROC}_1.} \tag{6}$$

Again taking the bilateral Laplace transform on both sides of Eq. (4):

$$\boxed{P_{yy}(s) = P_{xx}(s)H(s)H^*(-s), \quad s \in \mathbf{ROC}_2.} \tag{7}$$

Upon evaluating of the PSD $P_{yy}(s)$ on the imaginary axis this relation reduces to:

$$\boxed{P_{yy}(\Omega) = P_{xx}(\Omega)|H(\Omega)|^2, \quad \Omega \in \mathbf{R}^1.} \tag{8}$$

This relation is significant in that implies that the PSD of the output random signal is a spectrally shaped version of the PSD of the input. Since we are looking at bilateral transforms, it becomes necessary to associate a *region of convergence* (ROC) with the PSD to make the Laplace transformation unique.

Output Average Power

The average power of the output random signal can then be computed in the time-domain via:

$$E\{Y^2(t)\} = \sigma_y^2 = R_{yy}(0). \tag{9}$$

From a frequency-domain perspective we can evaluate the ouput average power via:

$$P_{\text{ave}}^y = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{xx}(\Omega)|H(\Omega)|^2 d\Omega. \tag{10}$$

This relation is significant from the viewpoint that the steady state average power of the output random process can be computed in either the time-domain or the frequency-domain. In the time domain it is evaluated as the ACF of the output process evaluated at the zero lag. In the frequency domain it is evaluated as the area under the output PSD.

Non-zero Means

If the input random signal under consideration were not a zero mean random process then the ACF in the analysis should be replaced with the auto-covariance function $C_{xx}(\tau)$ given by:

$$C_{xx}(\tau) = R_{xx}(\tau) - \mu_x^2. \tag{11}$$

where the contribution of the nonzero mean has been subtracted off. In this case X(t) can be written as

$$X(t) = X_o(t) + \mu_x, \tag{12}$$

where $X_o(t)$ is the zero mean part of $X(t)$. The ACF of the non-zero mean signal $X(t)$ can be written as :

$$R_{xx}(\tau) = R_{x_o x_o}(\tau) + \mu_x^2. \tag{13}$$

The PSD of the $X(t)$ can be written as:

$$P_{xx}(\Omega) = P_{x_o x_o}(\Omega) + 2\pi\mu_x^2\delta(\Omega). \tag{14}$$

Design of optimal filters will require, as we will see, rational power spectra and the presence of impulsive components will complicate things.