ECE - 541, FALL 2018

PROBABILITY THEORY & STOC. PROC.

Example: Scaling, Shifting, Modulation

Shifting

(a) Suppose $X(t,\omega)$ is a WSS process that is first subjected to a simple delay:

$$Y(t,\omega) = X(t-t_0,\omega)$$

Since the process is WSS

$$f_X(x;t) = f_X(x;t-t_0)$$

$$f_X(x_1,x_2;t_1,t_2) = f_{X_1,X_2}(x_1,x_2;t_1-t_0,t_2-t_0)$$

$$\Rightarrow \quad \mu_X(t) = \mu_X(t-t_0) = \mu_X$$

$$\Rightarrow \quad \sigma_X^2(t) = \sigma_X^2(t-t_0) = \sigma_X^2$$

$\therefore$ First-order statistics are invariant to a delay

$$R_{XX}(t_1,t_2) = R_{XX}(t_1-t_0,t_2-t_0) = R_{XX}(\tau)$$

Second-order statistics are invariant to a delay

(b) Scaling

Suppose $Z(t, \omega) = X(at, \omega)$

$$E\{X_{at}(\omega)\} = \mu_x(at)$$
$$= \mu_x$$

$$Var\{Z(t,\omega)\} = E\{Z_t^2(\omega)\} - \mu_x^2$$
$$= E\{X^2(at)\} - \mu_x^2$$
$$= \sigma_x^2$$

$\Rightarrow$ First-order statistics invariant

$$R_{zz}(t_1, t_2) = E\{Z(t_1) Z^*(t_2)\}$$
$$= E\{X(at_1) X^*(at_2)\}$$
$$= R_{xx}(a(t_1 - t_2)) = R_{xx}(a\tau)$$

$\Rightarrow$ Second-order statistics are scaled

$\Rightarrow$ $Z(t, \omega)$ is still WSS

Modulation:

$$W(t,\omega) = X(t,\omega)\, e^{j\Omega_c t}$$

$$\mu_W(t) = E\{W_t(\omega)\}$$
$$= E\{X_t(\omega)\, e^{j\Omega_c t}\}$$

$$= \mu_x\, e^{j\Omega_c t}$$

$$\sigma_W^2(t) = E\{|W(t)|^2\} - |\mu_W(t)|^2$$

$$= E\{|X(t,\omega)|^2\} - |\mu_x|^2$$
$$= \sigma_x^2(t) = \sigma_x^2$$

Mean is time-varying
Variance is invariant
$\Rightarrow$ First-order statistics are time-varying, process is not $n=1$ stationary

$$R_{WW}(t_1,t_2) = E\{W(t_1)W^*(t_2)\}$$

$$= E\{X(t_1)e^{j\Omega_c t_1} X^*(t_2) e^{-j\Omega_c t_2}\}$$

$$= R_{xx}(t_1,t_2)\, e^{j\Omega_c(t_1-t_2)}\}$$

$$R_{WW}(t_1,t_2) = R_{xx}(\tau)\, e^{j\Omega_c \tau}$$

$\Rightarrow$ $W(t)$ is in general not WSS because mean is time-varying

However if $\mu_x = 0$ then
it is both first-order and
second-order stationary $\Rightarrow$ WSS