Brownian Motion

$$m\ddot{x}(t) + f\,\dot{x}(t) + K\,x(t) = v(t),$$

where $v(t)$ is a WGN process with

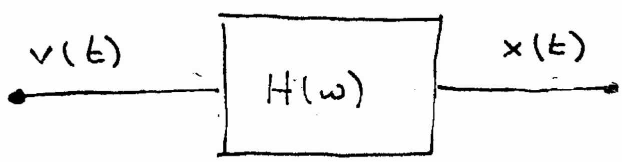
$$P_{vv}(\omega) = 2KTf \quad , \forall \omega \in \mathbb{R}^1 \quad (\text{zero mean})$$

f : Coefficient of friction

$v(t)$: Collision force

$Kx(t)$: Spring force

Power Spectral Analysis



$$H(s) = \frac{1}{ms^2 + fs + K}$$

$$P_{xx}(\omega) = P_{vv}(\omega)\,|H(\omega)|^2$$

$$P_{xx}(\omega) = P_{vv}(\omega)\left|\frac{1}{(K - m\omega^2) + j\omega f}\right|^2$$

$$P_{xx}(\omega) = \frac{\sigma_v^2}{\omega^2 f^2 + (k - m\omega^2)^2} \quad , \quad \omega \in \mathbb{R}^1$$

$$R_{xx}(\tau) = F^{-1}\left\{\frac{\sigma_v^2}{\omega^2 f^2 + (K - m\omega^2)^2}\right\}$$

$$= F^{-1}\left\{\frac{\sigma_v^2}{|K - m\omega^2 + j\omega f|^2}\right\}$$

$$R_{xx}(\tau) = \mathcal{L}^{-1}\left\{ \frac{\sigma_v^2}{T(s)\,T(-s)} \right\},$$

where $T(s) = ms^2 + fs + K$

$$P_{xx}(s) = \frac{\dfrac{\sigma_v^2}{m^2}}{\left(s^2 + \frac{f}{m}s + \frac{K}{m}\right)\left(s^2 - \frac{f}{m}s + \frac{K}{m}\right)}$$

$$S_{1,2} = -\frac{f}{2m} \pm \sqrt{\left(\frac{f}{2m}\right)^2 - \frac{K}{m}}$$

$$\alpha = \frac{f}{2m}$$

$$\omega_o^2 = \frac{K}{m}$$

$$-\beta^2 + \alpha^2 = \omega_o^2$$

For complex roots $S_{1,2} = -\alpha \pm j\beta$

$$f^2 - 4Km < 0 \quad \text{or} \quad \alpha^2 < \omega_o^2$$

(Critical Damping)

$$P_{xx}(s) = \frac{K_1 + K_2 s}{s^2 + 2\alpha s + \omega_0^2} + \frac{K_3 + K_4 s}{s^2 - 2\alpha s + \omega_0^2}$$

$$= \frac{\sigma_v^2 / m^2}{(s^2 + 2\alpha s + \omega_0^2)(s^2 - 2\alpha s + \omega_0^2)}$$

$$K_2 = \frac{\sigma_v^2}{4 \alpha \omega_0^2 m^2}, \qquad K_1 = \frac{\sigma_v^2}{2 m^2 \omega_0^2}$$

$$K_4 = -\frac{\sigma_v^2}{4 \alpha m^2 \omega_0^2}, \qquad K_3 = +\frac{\sigma_v^2}{2 m^2 \omega_0^2}$$

$$P_{xx}(s) = \frac{\sigma_v^2 / m^2}{4 \alpha \omega_0^2} \left(\frac{s + 2\alpha}{s^2 + 2\alpha s + \omega_0^2} + \frac{2\alpha - s}{s^2 - 2\alpha s + \omega_0^2} \right)$$

$$R_{xx}(\tau) = \frac{\sigma_v^2 / m^2}{4 \alpha \omega_0^2} e^{-|\tau| \cdot \alpha} \left(\cos \beta\tau + \frac{\alpha}{\beta} \sin(|\tau| \beta) \right)$$

$$= \frac{\sigma_v^2 / m^2}{2 f K / m^2} e^{-|\tau| \alpha} \left(\cos(\beta\tau) + \frac{\alpha}{\beta} \sin(\beta |\tau|) \right)$$

$$= \frac{2 K_B T f}{2 f K} e^{-\alpha |\tau|} \left\{ \cos(\beta\tau) + \frac{\alpha}{\beta} \sin(\beta |\tau|) \right\}$$

$$= \frac{K_B T}{K} e^{-\alpha |\tau|} \left\{ \cos(\beta\tau) + \frac{\alpha}{\beta} \sin(\beta |\tau|) \right\}$$

$$R_{xx}[0] = P_{ave}^{x} = \frac{K\,T}{k}$$

K : Maxwells - Boltzman Constant

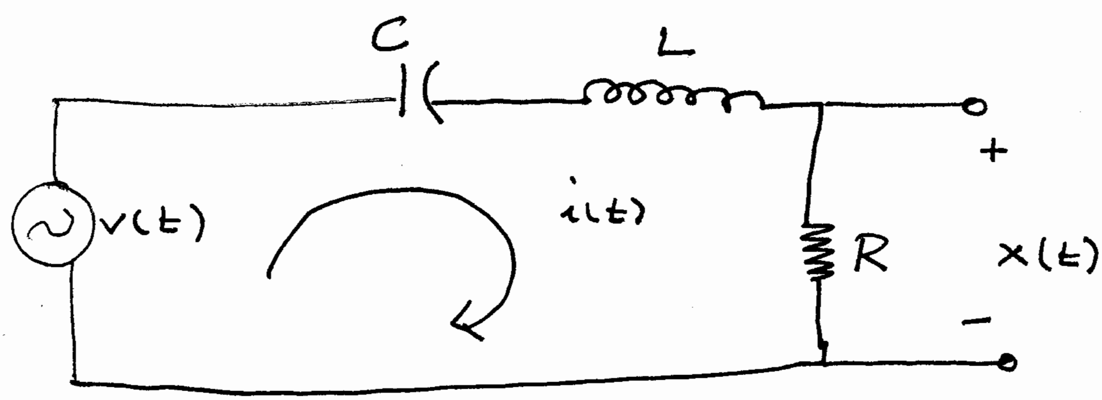
T : Temperature in kelvin

k : Spring constant of external force

$$f_{x(t)}(x;t) \approx N\left(0, \frac{KT}{k}\right)$$

Equivalent Circuit Model

Thermal Noise Model:

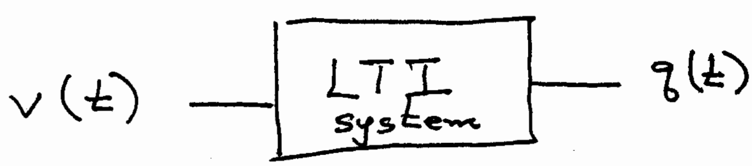


$$C v_c(t) = q(t) \qquad \frac{dq}{dt} = i(t)$$

$$V_c(t) = \frac{1}{C} q(t)$$

$$v(t) = \frac{1}{C} q(t) + R \frac{dq}{dt} + L \frac{d^2 q}{dt^2}$$

$$L \ddot{q}(t) + R \dot{q}(t) + \frac{1}{C} q(t) = v(t)$$



$$T(s) = \frac{1}{s^2 L + sR + \frac{1}{C}}$$

$$T(s) = \frac{C}{s^2 LC + sRC + 1}$$

$$T(s) = \frac{1/L}{s^2 + s\frac{R}{L} + 1/LC}$$

(Lowpass - filtered WGN)