ECE - ... Fall 2010

Example : Classification

$$Y(n) = \sum_{i=1}^{n} X(i) \, , \quad X(n) \text{ are}$$

i.i.d R.V's with PDF $f_X(x)$

(1) $$F_{Y(n)}\big|_{Y(n-1)=y_{n-1},\, Y(n-2)=y_{n-2} \cdots\cdots}$$

$$\triangleq P_r\{Y(n) \le y \,|_{Y(n-1)=y_{n-1},\, Y(n-2)=y_{n-2}, \cdots\cdots}\}$$

$$= P_r\{Y(n-1) + X(n) \le y \,|_{Y(n-1)=y_{n-1},\, Y(n-2)=y_{n-2}, \cdots}\}$$

$$= P_r\{X(n) \le y_n - y_{n-1} \,|\, Y(n-1)=y_{n-1},\, Y(n-2)=y_{n-2}, \cdots\}$$

$$= P_r\{X(n) \le y_n - y_{n-1} \,|\, Y(n-1) = y_{n-1}\}$$

$$= P_r\{X(n) + y_{n-1} \le y \,/\, Y(n-1) = y_{n-1}\}$$

$$= P_r\{X(n) + Y(n-1) \le y \,/\, Y(n-1) = y_{n-1}\}$$

$$= P_r\{Y(n) \le y \,/\, Y(n-1) = y_{n-1}\}$$

$$= F_{Y(n)}/_{Y(n-1)=y_{n-1}}$$

$\Rightarrow$ $Y(n)$ is a Markov random sequence

(2) $E\{Y(n) \mid Y(n-1) = y_{n-1}, Y(n-2) = y_{n-2}, \ldots\}$

$= E\{Y(n-1) + X(n) \mid Y(n-1) = y_{n-1}, Y(n-2) = y_{n-2}, \ldots\}$

$= E\{Y(n-1) \mid_{P.H} Y(n)\} + E\{X(n) \mid_{P.H} Y(n)\}$

$= y_{n-1} + E\{X(n)\}$

($X(n)$ is independent of $Y(n-1), \ldots$)

$E\{Y(n) \mid Y(n-1) = y_{n-1}, Y(n-2) = y_{n-2}, \ldots\}$

$= y_{n-1} + 0 = y_{n-1}$

$\Rightarrow$ $Y(n)$ is a Martingale

(3) $Y(1) = X(1)$

$Y(2) = X(1) + X(2)$

$\vdots$

$Y(n) = X(1) + X(2) + \ldots X_n$

In matrix format :

$$\begin{pmatrix} Y(1) \\ \vdots \\ Y(n) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & 1 & 1 & 0 & \cdots & & 0 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}_{A} \begin{pmatrix} X(1) \\ X(2) \\ \vdots \\ X(n) \end{pmatrix}$$

$$\det(A) = 1$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ -1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

$$f_{\underline{Y}}(\underline{y}) = \frac{1}{|\det(A)|} f_{\underline{x}}(A^{-1}\underline{y})$$

$$= f_{\underline{x}}(y_1, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1})$$

If $X(i)$ are i.i.d with PDF $f_X(x)$

$$f_{\underline{Y}}(y) = f_X(y_1)\, f_X(y_2 - y_1)\, f_X(y_3 - y_2) \cdots f_X(y_n - y_{n-1})$$

$$f_{Y(n)/Y(n-1)=y_{n-1}}$$

$$\triangleq f_{Y(n)/Y(n-1)=y_{n-1},\,Y(n-2)=y_{n-2},\,\ldots\ldots\,Y(1)=y_1}$$

$$= \frac{f_{\underline{Y}}\left(Y(n)=y_n,\,Y(n-1)=y_{n-1},\,\ldots\,Y(1)=y_1\right)}{f_{\underline{Y}}\left(Y(n-1)=y_{n-1},\,Y(n-2)=y_{n-2},\,\ldots\,Y(1)=y_1\right)}$$

$$= \frac{\cancel{f_X(y_1)}\,\cancel{f_X(y_2-y_1)}\,\cancel{f_X(y_3-y_2)}\cdots\cdots f_X(y_n-y_{n-1})}{\cancel{f_X(y_1)}\,\cancel{f_X(y_2-y_1)}\,\cancel{\cdots}\cdots\cdots\,\cancel{f_X(y_{n-1}-y_{n-2})}}$$

$$= f_X(y_n - y_{n-1})$$