On Mean Squared Convergence

A concept that is central to the notion of metric spaces and also to any discussion on modes of convergence of random variables, that we will look at in detail in this course, is the notion of convergence of a sequence. A deterministic sequence $\{x_n\}, n = 1, 2, \ldots$ is said to converge to a limit point x if and only if for every $\epsilon > 0$, $\exists$ an integer $N(\epsilon) \ni$:

$$\lim_{n\to\infty} |x_n - x| < \epsilon, \quad n > N(\epsilon).$$

This criteria is sometimes refered to as the *cauchy convergence criteria* (CCC). In the event that the limit point x of the sequence is not known the CCC can be modified as saying that the sequence $\{x_n\}, n = 1, 2, \ldots$ converges if and only if for any $m > 0$:

$$\lim_{n\to\infty} |x_{n+m} - x_n| = 0.$$

In the case of a sequence of random variables $\{X_n(\omega)\}, n = 1, 2, \ldots$ defined on a probability space $(\Omega, \mathcal{F}, P)$, this sequence is said to converge to the random variable X in the *mean squared sense* if and only if:

$$\lim_{n\to\infty} E\{|X_n(\omega) - X(\omega)|^2\} = 0.$$

The equivalent statement for mean-squared convergence when the limit point $X(\omega)$ is not known is that the sequence $\{X_n(\omega)\}, n = 1, 2, \ldots$ converges in the mean-squared sense if and only if for any $m > 0$:

$$\lim_{n\to\infty} E\{|X_{n+m}(\omega) - X_n(\omega)|^2\} = 0.$$

From our earlier discussion of a Hilbert space of random variables with finite second moment and inner product defined via:

$$< X, Y >= E\{XY\},$$

we saw that random variables that belong to this space obey the C-S inequality given by:

$$E^2\{XY\} \leq E\{X^2\}E\{Y^2\}.$$

Specifically if we choose $X = X_n(\omega) - X(\omega)$ and $Y = 1$, we obtain the inequality:

$$E^2\{(X_n(\omega) - X(\omega)\} \leq E\{(X_n(\omega) - X(\omega))^2\}.$$

Proceeding to the limit as $n \to \infty$ we have:

$$\lim_{n\to\infty} E^2\{X_n(\omega) - X(\omega)\} = \lim_{n\to\infty} E\{(X_n(\omega) - X(\omega))^2\} = 0.$$

Consequently if the sequence $\{X_n(\omega)\}, n = 1, 2, \ldots$ converges to the random variable $X(\omega)$ in the mean-squared sense then:

$$\lim_{n\to\infty} E\{X_n(\omega)\} = E\{X(\omega)\}$$

This statement implies that if we are talking about limits in the mean-squared sense then the operations of expectation and limit can be interchanged, i.e., they commute. Let us now go one step further and look at random sequences $\{X_n(\omega)\}$ and $\{Y_n(\omega)\}$ and $V_n(\omega)$ that converge in the mean-squared sense to the random variables $X(\omega)$ and $Y(\omega)$ and $V(\omega)$. From the parallelogram law we have that for fixed constants a and b:

$$\begin{aligned} E\{(aX_n(\omega) + bY_n(\omega) - aX(\omega) - bY(\omega))^2\} &\leq 2a^2 E\{(X_n(\omega) - X(\omega))^2\} \\ &+ 2b^2 E\{(Y_n(\omega) - Y(\omega))^2\} \end{aligned}$$

Proceeding to the limit $n \to \infty$ as we did before we see that the sequence of random variables $aX_n(\omega) + bY_n(\omega)$ converges in the mean-squared sense to the random variable $aX(\omega) + bY(\omega)$, i.e., the operation of limit in the mean-squared sense is linear.