On Modes of Convergence of Random Sequences

Consider again the notion of a probability space and the underlying triplet $(\Omega, \mathcal{F}, P)$. Our goal is to clarify the notion of convergence of a sequence of random variables on this space. Of course the notion of convergence on this space can be with respect to different parameters on this space. If we are concerned about moments of the sequence then the mode of interest is convergence in the *mean-squared sense*. If we are looking at the progression of things in terms of the induced distribution then the mode of interest is convergence in the *distribution sense*, and when we are interested in the probability measure being assigned to the various events in the underlying σ-field then the mode of interest is convergence in the *probability measure*.

A random sequence $X_n(\omega), n \in \mathbf{I}$ is said to converge *almost surely* to the random variable $X(\omega)$ if for every $\omega \in \Omega$, the sequence of numbers $X_n(\omega)$ converges. This limit in general will be dependent on the choice of the outcome ω and is in general a random variable and and will be denoted as $X(\omega)$, i.e.,

$$\Pr(\lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1$$

Alternatively if we denote the set $A = \{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$ then $\Pr(A) = 1$ or $\Pr(A^c) = 0$. This mode of convergence is the strongest mode of convergence since it requires convergence of the sequence $X_n(\omega)$ to $X(\omega)$ on every sample function except on those that have zero probability measure assignment. This mode of convergence is sometimes referred to as *convergence with probability 1.*

A random sequence $X_n(\omega), n \in \mathbf{I}$ is said to converge *in the mean-squared sense* to the random variable $X(\omega)$ if:

$$\lim_{n\to\infty} E\{|X_n(\omega) - X(\omega)|^2\} = 0.$$

This mode of convergence deals with the moments of the random variables and amounts to requiring that the average power associated with the error $|X_n(\omega) - X(\omega)|$ converge to zero. This mode of convergence is of practical importance and will be used extensively to deal with concepts of continuity, differentiability and integrability of stochastic processes.

A random sequence $X_n(\omega)$ is said to converge in *probability measure* to the random variable $X(\omega)$ if for every $\epsilon > 0$ we have:

$$\lim_{n\to\infty} \Pr(|X_n(\omega) - X(\omega)| > \epsilon) = 0.$$

Note here that we are not placing any constraints that this hold for every outcome in Ω. What is of concern is that random variable $X_n(\omega)$ for a specific n converge to $X(\omega)$. Alternatively we can view convergence in probability as the convergence of the probability measure assigned to the event $|X_n(\omega) - X(\omega)|$ to zero as n gets large.

A random sequence $X_n(\omega)$ is said to converge in the *distribution sense* to the random variable $X(\omega)$ if:

$$\lim_{n\to\infty} \Pr(\{X_n(\omega) \leq x\}) = \Pr(\{X(\omega) \leq x\}).$$

In other words, the distribution induced by the event $\{X_n(\omega) \leq x\}$ converges to the distribution of X, i.e.,

$$\lim_{n\to\infty} F_n(x) = F(x).$$

This mode of convergence is the weakest form of convergence and in essence says that the CDF's converge.