Example: Convergence of Sequences of Random Variables

Consider a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, 1]$ and P is the uniform probability law on Ω. Suppose we define a sequence of random variables on this space via:

$$X_n(\omega) = \begin{cases} n\omega, & \omega \in [0, \frac{1}{n}] \\ 0, & \omega \in [\frac{1}{n}, 1] \end{cases}$$

Convergence in the almost sure sense: For any $\omega_o \in \Omega$, $X_n(\omega_o) = 0,\ n > N(\omega_o)$, because the support for the sequence is shrinking. This implies that the sequence of random variables $X_n(\omega)$ coverges almost surely to the random variable, $X(\omega) = 0$ at most points except at the point $\omega = 0$, since:

$$A \equiv \{\omega \ni \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\} = \{\omega = 0\},\ \Pr(A) = 0.$$

Convergence in the MS sense: Let us now calculate the variances of the sequence of random variables. Using the uniform probability law:

$$\lim_{n\to\infty} E\{X_n^2(\omega)\} = \lim_{n\to\infty} \int_0^{\frac{1}{n}} n^2\omega^2 dP = \lim_{n\to\infty} \frac{1}{3n} = 0.$$

Consequently the sequence of random variables $X_n(\omega)$ converges in the MS sense to the random variable $X(\omega) = 0$.
Convergence in probability: By the Chebyshev inequality, for any $\epsilon > 0$:

$$0 \leq \lim_{n\to\infty} \Pr(|X_n(\omega)| > \epsilon) \leq \lim_{n\to\infty} \frac{E\{X_n^2(\omega)\}}{\epsilon^2} = 0.$$

Consequently the sequence of random variables $X_n(\omega)$ converges in probability to the random variable $X(\omega) = 0$.
Convergence in distribution: Let us now look at the CDF of the sequence of random variables:

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1-x}{n} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Consequently:

$$\lim_{n\to\infty} F_{X_n}(x) = u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

As can be seen, the sequence of random variables $X_n(\omega)$ converges to the random variable $X(\omega) = 0$ in distribution. Furthermore in this particular example the sequence $X_n(\omega)$ converges in all four senses to the random variable $X(\omega) = 0$.