2 Cyclostationarity

In the determinitic case, a signal is called periodic if it repeats after a period of time. In this section we will look at the statistical equivalent of periodicity. A random process $X(t)$ is said to be n^{th}-order cyclostationary if the n^{th}-order PDF is periodic:

$$f_{\mathbf{X}}(\mathbf{x};\mathbf{t}) = f_{\mathbf{X}}(\mathbf{x};\mathbf{t}+T_o^{(n)}\mathbf{1})\ ,\ T_o^{(n)} \in \mathbf{R}. \tag{5}$$

The periodicity of the statistics, $T_o^{(n)}$ is often referred to as the *cyclostationarity parameter*.

As in the case of stationarity, $n = 1$ cyclostationarity implies that

$$\begin{aligned} \mu_X(t) &= \mu_X(t+T_o^{(1)}) \\ \sigma_X(t) &= \sigma_X(t+T_o^{(1)}) \end{aligned} \tag{6}$$

Second-order cyclostationarity specifically implies that

$$R_{XX}(t_1,t_2) = R_{XX}(t_1+T_o^{(2)}, t_2+T_o^{(2)}). \tag{7}$$

Since the ACF of the process is dependent on both t_1 & t_2, these processes are non-stationary. Furthermore the ensemble ACF is periodic and as a consequence it has a Fourier series expansion of the form:

$$R_{XX}(t,\tau) = E\left[X(t)X^*(t-\tau)\right] = \sum_{n=-\infty}^{\infty} \tilde{R}_{XX}^{(n)}(\tau)\exp\left(j\frac{2\pi}{T_o^{(2)}}nt\right) \tag{8}$$

The Fourier series coefficients $R_{xx}^{(n)}(\tau)$ are referred to as the *cyclic autocorrelation function*. The DC Fourier coefficient in particular is called the *time averaged ACF* and is determined via:

$$\tilde{R}_{XX}^{(o)}(\tau) = \left(\frac{1}{T_o^{(2)}}\right)\int_{-\frac{T_o^{(2)}}{2}}^{\frac{T_o^{(2)}}{2}} R_{XX}(t,t-\tau)dt. \tag{9}$$

The corresponding time-averaged power spectrum is defined via the Fourier transform:

$$P_{xx}^{(o)}(\Omega) = \int_{-\infty}^{\infty} R_{xx}^{(o)}(\tau)\exp\left(-j\Omega\tau\right)d\tau$$

Using the cyclic autocorrelation function we can define a corresponding cyclic power-spectrum $P_{xx}^{(n)}(\Omega)$ via:

$$\begin{aligned} P_{xx}^{(n)}(\Omega) &= \int_{-\infty}^{\infty} R_{xx}^{(n)}(\tau)\exp\left(-j\Omega\tau\right)d\tau \\ R_{xx}^{(n)}(\tau) &= \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} P_{xx}^{(n)}(\Omega)\exp\left(j\Omega\tau\right)d\Omega. \end{aligned} \tag{10}$$

The motivation behind the definition of the time-averaged ACF is to average out and remove the dependence of the ACF on the t variable so that we do not have to deal with two dimensional Fourier transforms.