Example: cyclostationary process

Consider a random process $X(t)$ that is defined via the relation:

$$X(t,\omega) = A(\omega)\cos(\omega_o t) + B(\omega)\sin(\omega_o t), \quad t \in \mathbf{R}, \quad \omega \in \mathbf{R},$$

where $A \sim N(\mu_1, \sigma_1^2)$ and $B \sim N(\mu_2, \sigma_2^2)$ and are independent of each other. If we now sample this process at any $t_o \in \mathbf{R}$ we obtain the random variable:

$$X(t_o,\omega) = X(\omega) = A(\omega)\cos(\omega_o t_o) + B(\omega)\sin(\omega_o t_o).$$

Since both A and B are Gaussian random variables, the random process $X(\omega)$ has Gaussian statistics with a mean given by:

$$\mu_x(t) = \mu_1 \cos(\omega_o t) + \mu_2 \sin(\omega_o t).$$

The corresponding variance of the process $X(t)$ is given by:

$$\sigma_x^2(t) = E\left\{(A\cos(\omega_o t) + B\sin(\omega_o t))^2\right\} - (\mu_1\cos(\omega_o t) + \mu_2\sin(\omega_o t))^2.$$

Using the fact that A and B are independent random variables we can simply the above expression to:

$$\sigma_x^2(t) = \sigma_1^2\cos^2(\omega_o t) + \sigma_2^2\sin^2(\omega_o t)$$

Since the mean and the variance of this process are both periodic in the t variable with a period $T_o^{(1)} = \frac{2\pi}{\omega_o}$ this process is first-order cyclostationary. If we now sample the process at two different instants $t_1, t_2 \in \mathbf{R}$ we obtain the random variables X_1 and X_2 defined via:

$$\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) = \underbrace{\left(\begin{array}{cc} \cos(\omega_o t_1) & \sin(\omega_o t_1) \\ \cos(\omega_o t_2) & \sin(\omega_o t_2) \end{array}\right)}_{\mathbf{M}} \left(\begin{array}{c} A \\ B \end{array}\right).$$

The mean vector and the covariance matrix associated with this Gaussian random process is then given by:

$$\mu_x = \mathbf{M}\left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array}\right), \quad \mathbf{C}_{xx} = \mathbf{M}\ \mathrm{diag}(\sigma_1^2, \sigma_2^2)\ \mathbf{M}^T.$$

The auto-covariance function associated with this process is obtained as the off-diagonal element of the covariance matrix $\mathbf{C}_{xx}$:

$$C_{xx}(t_1,t_2) = \frac{\sigma_1^2 + \sigma_2^2}{2}\cos(\omega_c(t_2 - t_1)) + \frac{\sigma_1^2 - \sigma_2^2}{2}\cos(\omega_c(t_1 + t_2)).$$

Substituting $t_1 = t$ and $t_2 = t - \tau$ we obtain:

$$C_{xx}(t,\tau) = \frac{\sigma_1^2 + \sigma_2^2}{2}\cos(\omega_c\tau) + \frac{\sigma_1^2 - \sigma_2^2}{2}\cos(2\omega_c t - \omega_c\tau).$$

This expression is periodic in the t variable with a fundamental period of $T_o^{(2)} = \frac{2\pi}{2\omega_o}$. Consequently all other second-order statistics of the process such as $R_{xx}(t,\tau)$ and $\rho_{xx}(t,\tau)$ are also periodic with the same periodicity and the random process $X(t)$ is second-order cyclostationary. The Gaussian statistics make this process strict sense cyclostationary.

The time-averaged auto-covariance function obtained from averaging out the dependence on the t variable is given by:

$$\tilde{C}_{xx}(\tau) = \int_{-T_o^{(2)}/2}^{T_o^{(2)}/2} C_{xx}(t,\tau)dt = \frac{\sigma_1^2 + \sigma_2^2}{2}\cos(\omega_c\tau).$$

We will revisit this specific form again when we look at the Rice representation for bandpass processes. It suffices to say that in general the process $X(t)$ is not stationary but cyclostationary. For the specific case where $\mu_1 = \mu_2 = 0$ and $\sigma_2^2 = \sigma_2^2 = \sigma^2$ we obtain the special case where $\mu_x(t) = 0$, $\sigma_x^2(t) = \sigma^2$ and $R_{xx}(\tau) = \sigma^2\cos(\omega_o\tau)$. In this case, the process $X(t)$ becomes WSS and because the process is Gaussian it is also SSS.