Optimal Wiener Deconvolution

The Wiener deconvolution problem seeks to extract an estimate of the SOI $d[n]$ from observations of the form:

$$x[n] = \sum_{k=-\infty}^{\infty} g[k]d[n-k] + \epsilon[n],$$

where $g[n]$ is the distortion system and $\epsilon[n]$ is white, additive observation noise that is uncorrelated with the SOI. The cross-correlation between the SOI and the observations can be computed as:

$$r_{dx}[k] = E\left\{d[n]\left\{\sum_{p=-\infty}^{\infty} g[p]d[n-k-p] + \epsilon[n-k]\right\}\right\} = \sum_{p=-\infty}^{\infty} g[p]r_{dd}[k+p] = g[-k] * r_{dd}[k]$$

Taking the Zee transform on both sides:

$$P_{dx}(z) = G^*\left(\frac{1}{z^*}\right)P_{dd}(z).$$

In a similar fashion the PSD of the observation can be computed as:

$$P_{xx}(z) = G(z)G^*\left(\frac{1}{z^*}\right)P_{dd}(z) + \sigma_\epsilon^2.$$

If the inverse system corresponding to the distortion operator $g[n]$ exists then we can completely eliminate the distortion at the expense of ambient noise amplification:

$$\sigma_f^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_\epsilon^2}{|G(e^{j\omega})|^2} d\omega.$$

The system function of the optimal Wiener deconvolver that trades of elimination of the distortion with noise reduction is given by:

$$H_{\text{opt}}(z) = \frac{G^*\left(\frac{1}{z^*}\right)P_{dd}(z)}{G(z)G^*\left(\frac{1}{z^*}\right)P_{dd}(z) + \sigma_\epsilon^2}.$$

This expression can be factorized into two parts:

$$H_{\text{opt}}(z) = \left(\frac{1}{G(z)}\right)\left(\frac{P_{dd}(z)}{P_{dd}(z) + \frac{\sigma_\epsilon^2}{G(z)G^*(1/z^*)}}\right).$$

Consequently the optimal Wiener deconvolver can be considered as a cascade of the inverse filter, provided that it exists, and a Wiener smoother than minimizes the effect of noise amplification from the inverse filtering operation. The MMSE associated with this optimal Wiener deconvolver is given by:

$$\epsilon_{\min}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{dd}(e^{j\omega})d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|G(e^{j\omega})|^2(P_{dd}(e^{j\omega}))^2}{|G(e^{j\omega})|^2 P_{dd}(e^{j\omega}) + \sigma_\epsilon^2} d\omega.$$

Upon simplification this expression becomes:

$$\epsilon_{\min}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_\epsilon^2 P_{dd}(e^{j\omega})}{|G(e^{j\omega})|^2 P_{dd}(e^{j\omega}) + \sigma_\epsilon^2} d\omega.$$