3 Ergodic Processes

In the event that the distributions and statistics are not available we can avail ourselves of the time averages from the particular sample function. The mean of the sample function $X_{\lambda_o}(t)$ is referred to as the *sample mean* of the process $X(t)$ and is defined via :

$$\langle \mu_X \rangle_T = \left(\frac{1}{T}\right) \int_{-\frac{T}{2}}^{\frac{T}{2}} X_{\lambda_o}(t) dt. \tag{26}$$

This quantity is actually a random-variable by itself because its value depends on the particular sample function over which it was calculated. The *sample variance* of the random process is defined similarly via

$$\left\langle \sigma_X^2 \right\rangle_T = \left(\frac{1}{T}\right) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| X_{\lambda_o}(t) - \langle \mu_X \rangle_T \right|^2 dt. \tag{27}$$

The time-averaged sample ACF is obtained via the relation:

$$\langle R_{XX} \rangle_T = \left(\frac{1}{T}\right) \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x^*(t-\tau) dt. \tag{28}$$

These quantities are in general not the same as the ensemble averages described before. A random process $X(t)$ is said to be *ergodic* in the mean, i.e., *first-order ergodic* if the mean of sample average asymptotically approaches the ensemble mean:

$$\begin{aligned} \lim_{T\to\infty} E\left\{\langle \mu_X \rangle_T\right\} &= \mu_X(t) \\ \lim_{T\to\infty} \operatorname{Var}\left\{\langle \mu_X \rangle_T\right\} &= 0. \end{aligned} \tag{29}$$

In a similar sense a random process $X(t)$ is said to be *ergodic* in the ACF, i.e, *second-order ergodic* if :

$$\begin{aligned} \lim_{T\to\infty} E\left\{\langle R_{XX}(\tau) \rangle_T\right\} &= R_{XX}(\tau) \\ \lim_{T\to\infty} \operatorname{Var}\left\{\langle R_{XX}(\tau) \rangle_T\right\} &= 0. \end{aligned} \tag{30}$$

These conditions can be recognized as the requirements for convergence in the mean and mean-square convergence. In other words for a random signal to be ergodic in the particular parameter we require that the time-average estimate of the parameter be mean-square convergent to the actual ensemble average of the parameter. Second ergodicity is sometimes called as *wide sense ergodicity.*

The concept of ergodicity is also significant from a measurement perspective because in practical situations we do not have access to all the sample realizations of a random process. We therefore have to be content in these situations with the time-averages that we obtain from a single realization. Ergodic processes are signals for which measurements based on a single sample function are sufficient to determine the ensemble statistics. Random signals for which this property does not hold are referred to as *non-ergodic* processes. As before the Gaussian random signal is a an exception where strict sense ergodicity implies wide sense ergodicity.