ECE – 541

Ergodicity in ACF Example

$$\hat{R}_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \cos(\omega_c t + \theta_0)\cos(\omega_c t + \omega_c \tau + \theta_0)\,dt$$

$$\hat{R}_{xx}(\tau) =$$

$$\frac{1}{2T}\int_{-T/2}^{T/2} \cos(2\omega_c t - \omega_c \tau + 2\theta_0)\,dt$$

$$+ \frac{1}{2T}\int_{-T/2}^{T/2} \cos(\omega_c \tau)\,dt$$

$$= R_{xx}(\tau) + \frac{1}{2T}\int_{-T/2}^{T/2} \cos(2\omega_c t - \omega_c \tau + 2\theta_0)\,dt$$

$$\hat{R}_{xx}(\tau) - R_{xx}(\tau) = \frac{1}{2T}\int_{-T/2}^{T/2} \cos(2\omega_c t - \omega_c \tau + 2\theta_0)\,dt$$

For any realization of the process $X(t)$

NOTAS

$$E\left\{\hat{R}_{xx}(\tau)\right\}$$

$$= R_{xx}(\tau) + \underset{\theta_0}{E}\left\{\frac{1}{2T}\int_{-T/2}^{T/2}\cos(2\omega_c t - \omega_c\tau + 2\theta_0)\,dt\right\}$$

$$= R_{xx}(\tau) + \frac{1}{2T}\int_{-T/2}^{T/2}\cancelto{0}{\underset{\theta_0}{E}\left(\cos(2\omega_c t - \omega_c\tau + 2\theta_0)\right)}\,dt$$

$$\Longrightarrow \underset{\theta_0}{E}\left\{\hat{R}_{xx}(\tau)\right\} = R_{xx}(\tau)$$

and $\therefore$

$$\lim_{T\to\infty}\underset{\theta_0}{E}\left\{\hat{R}_{xx}(\tau)\right\} = R_{xx}(\tau)$$

$\Rightarrow$ $\hat{R}_{xx}(\tau)$ is an unbiased estimator of $R_{xx}(\tau)$

$$\left[\hat{R}_{xx}(\tau) - R_{xx}(\tau)\right]^2$$

$$= \left[\frac{1}{2T}\int_{-T/2}^{T/2} \cos(2\omega_c t - \omega_c \tau + 2\Theta_0)\, dt_1\right]$$

$$\left[\frac{1}{2T}\int_{-T/2}^{T/2} \cos(2\omega_c t_2 - \omega_c \tau + 2\Theta_0)\, dt_2\right]$$

$$= \frac{1}{4T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \cos(2\omega_c t_1 - \omega_c \tau + 2\Theta_0)\, dt_1 \cos(2\omega_c t_2 - \omega_c \tau + 2\Theta_0)\, dt_2$$

$$E_{\Theta_0}\left\{\left(\hat{R}_{xx}(\tau) - R_{xx}(\tau)\right)^2\right\}$$

$$= E_{\Theta_0}\left\{\frac{1}{4T^2}\int_{t_1}\int_{t_2} \cos(2\omega_c t_1 - \omega_c \tau + 2\Theta_0) \cos(2\omega_c t_2 - \omega_c \tau + 2\Theta_0)\, dt_1\, dt_2\right\}$$

$$= \frac{1}{4T^2}\iint E_{\Theta_0}\left\{\cos(2\omega_c t_1 - \omega_c \tau + 2\Theta_0) \cos(2\omega_c t_2 - \omega_c \tau + 2\Theta_0)\right\} dt_1\, dt_2$$

NOTAS

$$E_{\Theta_0}\left\{\cos(2\omega_c t_1 - \omega_c \tau + 2\Theta_0)\cos(2\omega_c t_2 - \omega_c \tau_0 + 2\Theta_0)\right\}$$

$$= \frac{1}{2} E_{\Theta_0}\left\{\underbrace{\cos(2\omega_c(t_1+t_2) - 2\omega_c \tau + 4\Theta_0)}_{\to 0} + \cos(2\omega_c(t_1-t_2))\right\}$$

$$= \frac{1}{2}\cos(2\omega_c(t_1-t_2))$$

$$E_{\Theta_0}\left\{\left[\hat{R}_{xx}(\tau) - R_{xx}(\tau)\right]^2\right\}$$

$$= \frac{1}{4T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \frac{1}{2}\cos(2\omega_c(t_1-t_2))\, dt_1\, dt_2$$

$$= \frac{1}{8T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \cos(2\omega_c \tau)\, dt_1\, dt_2$$

$$= \frac{1}{8T^2}\int_{-T}^{T}(T-|\tau|)\cos(2\omega_c\tau)\,d\tau$$

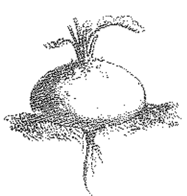
$$= \frac{1}{8T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\cos(2\omega_c\tau)\,d\tau$$

$$\lim_{T\to\infty} \underset{\Theta}{E}\left\{\left(R_{xx}(\tau)-\hat{R}_{xx}(\tau)\right)^2\right\}$$

$$= \lim_{T\to\infty}\frac{1}{8T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\cos(2\omega_c\tau)\,d\tau$$

As was shown in class using Parseval's theorem

$$= \lim_{T\to\infty}\frac{1}{16\pi T}\int_{-\infty}^{\infty}\left[\pi\delta(\Omega+2\omega_c)+\pi\delta(\Omega-2\omega_c)\right]\cdot T\,Sa^2\left(\frac{\Omega T}{2}\right)d\Omega$$



$$= \lim_{T\to\infty} \frac{1}{16} Sa^2\left(2\omega_c \frac{T}{2}\right) + \frac{1}{16} Sa^2\left(-2\omega_c\frac{T}{2}\right)$$

$$= \lim_{T\to\infty} \frac{1}{8} Sa^2(\omega_c T)$$

$$= \lim_{T\to\infty} \frac{1}{8} \frac{\sin^2(\omega_c T)}{\omega_c^2 T^2}$$

$$= 0$$

$$\Rightarrow \lim_{T\to\infty} Var\{\hat{R}_{xx}(\tau)\} = 0$$

$\Rightarrow X(t,\omega) = \cos(\omega_c t + \Theta(\omega))$, $\Theta \sim U([-\pi,\pi])$ is ergodic in ACF