Example: Oscillator with Random Phase

Consider the output of a sinusoidal oscillator that has a random phase and amplitude of the form:

$$X(t) = \cos(\Omega_c t + \Theta),$$

where $\Theta \sim \mathbf{U}([0, 2\pi])$. Writing out the explicit dependence on the underlying sample space $\mathbf{S}$ the oscillator output can be written as

$$x(t, \Theta) = \cos(\Omega_c t + \Theta). \tag{31}$$

This random signal falls in the continuous-time, continuous parameter, and continuous amplitude category and is useful in modeling propagation phenomena such as multi-path fading.

The first order distribution of this process can be found by looking at the distribution of the R.V

$$X_t(\Theta) = \cos(\Theta + \theta_o),$$

where $\Omega_c t = \theta_o$ is a non random quantity. This can easily be shown via the derivative method shown in class to be of the form:

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \ |x| < 1. \tag{32}$$

Note that this distribution is dependent only on the set of values that the process takes and is independent of the particular sampling instant t and the constant phase offset θ_o.

If the second-order distribution is needed then we use the conditional distribution of $x(t_2)$ as in :

$$f_{x(t_1),x(t_2)}(x_1, x_2) = f_{x(t_2)}(x_2) f_{x(t_1)|x(t_2)}(x_1|x_2) \tag{33}$$

If the value of $x(t_2)$ is to be equal to x_2 then we require $\cos(\Theta + \Omega_c t_2) = x_2$. This can happen only when :

$$\begin{aligned}\Theta &= \cos^{-1}(x_2) - \Omega_c t_2 \quad \text{or} \\ \Theta &= 2\pi - \cos^{-1}(x_2) - \Omega_c t_2,\end{aligned} \tag{34}$$

where $0 \le \cos^{-1}(x_2) \le \pi$. All other possible solutions lie outside the desired interval $[0, 2\pi]$. Consequently the random process at $t = t_1$ can only take on the values:

$$\begin{aligned}x(t_1) &= \cos\left(\Omega_c t_1 + \cos^{-1}(x_2) - \Omega_c t_2\right) \quad \text{or} \\ x(t_1) &= \cos\left(\Omega_c t_1 - \cos^{-1}(x_2) - \Omega_c t_2\right)\end{aligned} \tag{35}$$

Thus the conditional distribution of $x(t_1)$ given that $x(t_2) = x_2$ is of the form:

$$\begin{aligned}f_{x(t_1)|x(t_2)}(x_1|x_2) &= \left(\frac{1}{2}\right)\delta\left(x_1 - \cos\left[\Omega_c t_1 + \cos^{-1}(x_2) - \Omega_c t_2\right]\right) \\ &+ \left(\frac{1}{2}\right)\delta\left(x_1 - \cos\left[\Omega_c t_1 - \cos^{-1}(x_2) - \Omega_c t_2\right]\right).\end{aligned} \tag{36}$$

Combining Eq. (32) and Eq. (36) we have:

$$\begin{aligned}f_{x(t_1),x(t_2)}(x_1, x_2) &= \left\{\frac{1}{2\pi\sqrt{1-x_2^2}}\right\}\delta\left(x_1 - \cos\left[\Omega_c t_1 + \cos^{-1}(x_2) - \Omega_c t_2\right]\right) \\ &+ \left\{\frac{1}{2\pi\sqrt{1-x_2^2}}\right\}\delta\left(x_1 - \cos\left[\Omega_c t_1 - \cos^{-1}(x_2) - \Omega_c t_2\right]\right\}.\end{aligned} \tag{37}$$

Note here that the second-order PDF depends only on the difference variable $\tau = t_1 - t_2$. Let us look at the first-order and second-order moments of the random process $X(t)$. The mean of the process is obtained by taking the expectation operator with respect to the random parameter Θ on both sides of Eq. (31) keeping in mind that the expectation integral is a linear operation:

$$\begin{aligned}\mu_X &= E_\Theta(x_t(\Theta)) = E_\Theta\left[\cos\left(\Omega_c t + \Theta\right)\right] \\ &= E_\Theta\left[\cos(\Omega_c t)\cos(\Theta) - \sin(\Omega_c t)\sin(\Theta)\right] \\ &= E_\Theta\left[\cos(\Theta)\right]\cos(\Omega_c t) - E_\Theta\left[\sin(\Theta)\right]\sin(\Omega_c t). \end{aligned} \tag{38}$$

Since the random parameter Θ is uniformly distributed, the above expression reduces to:

$$\mu_X = \cos\left(\Omega_c t\right)\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\cos(\theta)d\theta - \sin\left(\Omega_c t\right)\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\sin(\theta)d\theta = 0. \tag{39}$$

The variance of the random process $X(t)$ is obtained by via

$$\sigma_X^2 = E_\Theta\left[\left(x_t(\Theta) - \mu_X\right)^2\right] = E_\Theta\left(\left[x_t(\Theta)\right]^2\right) - \mu_X^2 \tag{40}$$

Substituting the mean of the process in the above expression we have:

$$\sigma_X^2 = \left(\frac{1}{2\pi}\right)\int_0^{2\pi}\cos^2\left(\Omega_c t + \theta\right)d\theta = \left(\frac{1}{2\pi}\right)\int_0^{2\pi}\left[\frac{1 + \cos\left(2\Omega_c t + 2\theta\right)}{2}\right]d\theta = \frac{1}{2} \tag{41}$$

This means that the average power of the random sinusoidal signal $X(t)$ is

$$P_{\text{ave}}^X = \sigma_X^2 = \frac{1}{2}.$$

Note that this is the same as the average power of a sinusoid where the phase is not random. Let us look at the statistics from the second-order distribution. The correlation between the R.Vs $x(t_1)$ and $x(t_2)$ denoted as $R_{XX}\left(t_1, t_2\right)$ is obtained via:

$$\begin{aligned}R_{XX}\left(t_1, t_2\right) &= E_\Theta\left[x(t_1)x(t_2)\right] = \int_0^{2\pi}\cos\left[\Omega_c t_1 + \theta\right]\cos\left[\Omega_c t_2 + \theta\right]d\theta \\ &= \left(\frac{1}{4\pi}\right)\int_0^{2\pi}\cos\left[\Omega_c(t_1 + t_2) + 2\theta\right]d\theta + \left(\frac{1}{4\pi}\right)\int_0^{2\pi}\cos\left[\Omega_c(t_1 - t_2)\right]d\theta \\ &= \left(\frac{1}{2}\right)\cos\left[\Omega_c(t_1 - t_2)\right]. \end{aligned} \tag{42}$$

The covariance of R.Vs $X(t_1)$ and $X(t_2)$ denoted $C_{XX}\left(t_1, t_2\right)$ is given by:

$$C_{XX}\left(t_1, t_2\right) = R_{xx}\left(t_1, t_2\right) - \mu_X(t_1)\mu_X(t_2) = \left(\frac{1}{2}\right)\cos\left[\Omega_c(t_1 - t_2)\right]. \tag{43}$$

The correlation coefficient of the R.Vs $X(t_1)$ and $X(t_2)$ denoted $\rho_{XX}\left(t_1, t_2\right)$ is:

$$\rho_{XX}\left(t_1, t_2\right) = \cos\left[\Omega_c(t_1 - t_2)\right]. \tag{44}$$

Looking at the mean and the variance of the random process $X(t)$ we can see that they are shift-invariant and consequently the process is first-order stationary. The ACF and other second-order statistics of the process are dependent only on the variable $\tau = t_1 - t_2$. The random process $X(t)$ is therefore a WSS process also. The ACF can then expressed in terms of the variable $\tau = t_1 - t_2$ as:

$$R_{XX}(\tau) = \left(\frac{1}{2}\right)\cos(\Omega_c\tau). \tag{45}$$

Let us now look at time averages of a single sample function or realization of the random process $X(t)$. The sample mean of the random process irrespective of the sample realization that we choose is:

$$\langle\mu_X\rangle_T = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left[\Omega_c t + \Theta\right] dt. \tag{46}$$

As $T \to \infty$ we have:

$$\lim_{T\to\infty} \langle\mu_X\rangle_T = 0. \tag{47}$$

The sample mean of the process is therefore independent of the particular ensemble waveform used to calculate the time-average, i.e., independent of the value of Θ for the realization. Consequently we have:

$$\begin{aligned}\lim_{T\to\infty} E\left\{\langle\mu_X\rangle_T\right\} &= \mu_X(t) = 0 \\ \lim_{T\to\infty} \mathrm{Var}\left\{\langle\mu_X\rangle_T\right\} &= 0.\end{aligned} \tag{48}$$

The random process $X(t)$ is therefore ergodic in the mean (first-order ergodic). Let us now look at the sample ACF of the random process $X(t)$. The sample ACF is again independent of the particular realization of the process as evident from :

$$\begin{aligned}\lim_{T\to\infty} \langle R_{XX}(\tau)\rangle_T &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left[\Omega_c t + \Theta\right]\cos\left[\Omega_c(t-\tau) + \Theta\right] dt \\ \lim_{T\to\infty} \langle R_{XX}(\tau)\rangle_T &= \lim_{T\to\infty}\frac{1}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left[2\Omega_c t - \Omega_c\tau + 2\Theta\right] dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\Omega_c\tau)dt \\ \lim_{T\to\infty} \langle R_{XX}(\tau)\rangle_T &= \left(\frac{1}{2}\right)\cos(\Omega_c\tau) = R_{XX}(\tau).\end{aligned} \tag{49}$$

The random process $X(t)$ is therefore ergodic in the ACF (second-order ergodic).

The power-spectrum of this random signal, i.e., the Fourier transform of the ensemble ACF can then be computed as :

$$P_{XX}(\Omega) = \frac{\pi}{2}\left[\delta(\Omega + \Omega_c) + \left[\delta(\Omega - \Omega_c)\right]\right]. \tag{50}$$

Note that this expression for the power spectrum is identical to the expression for the spectrum of a deterministic sinusoidal signal.