Example : Ergodicity

Consider a random oscillator with both random amplitude and phase :

$$X(t,\omega) = A(\omega)\cos(\omega_0 t + \Theta(\omega)),$$

where

$$\left.\begin{array}{l} A \sim N(A_0, \sigma_A^2) \\ \Theta \sim U([-\pi,\pi]) \end{array}\right\} A \text{ \& } \Theta \text{ independent}$$

Ergodicity in Mean:

$$\hat{\mu}_x^{(T)} = \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} A_0 \cos(\omega_0 t + \theta_0)\,dt$$

$$\hat{\mu}_x^{(T)} = \frac{A_0}{T} k(T), \quad \text{where}$$

$$k(T) = \begin{cases} 0, & T = \frac{2\pi}{\omega_0} r, \; r \in I \\ c(T), & T \neq \frac{2\pi}{\omega_0} r, \quad c(T) < T \end{cases}$$

$$\lim_{T\to\infty} \hat{\mu}_x^{(T)} = \lim_{T\to\infty} \frac{A_0}{T} k(T) = 0$$

This implies that independent of the sample realization $\hat{\mu}_x^{(T)}$ converges to 0

The ensemble mean in this case is

$$\mu_x = E\{X(t)\} = 0$$

$$\lim_{T \to \infty} E\{\hat{\mu}_x^{(T)}\} = \lim_{T\to\infty} E\{A_{os}\} \frac{k(T)}{T}$$

$$= E\{A_o\} \lim_{T\to\infty} \frac{k(T)}{T} = 0$$

$$\Longrightarrow \lim_{T\to\infty} E\{\hat{\mu}_x^{(T)}\} = \mu_x$$

$$\mathrm{Var}\{\hat{\mu}_x^{(T)}\} = E\{(\hat{\mu}_x^{(T)} - \mu_x)^2\}$$

$$= E\{[\hat{\mu}_x^{(T)}]^2\}$$

We require that $\lim_{T\to\infty} \mathrm{Var}\{\hat{\mu}_x^{(T)}\} = 0$

or $$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} R_{xx}(\tau)\left(1 - \frac{|\tau|}{T}\right) d\tau = 0$$

or $$\Rightarrow \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} R_{xx}(\tau) \Lambda_T(\tau)\, d\tau = 0$$

or $$\Rightarrow \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} E\left\{\frac{A_{os}^2}{2}\right\} \pi\left(\delta(\Omega - \omega_o) + \delta(\Omega + \omega_o)\right) T \,\mathrm{sinc}^2(\Omega T/2\pi)\, d\Omega = 0$$

or $$\Rightarrow \lim_{T\to\infty} \frac{1}{\cancel{2\pi T}} E\left\{\frac{A_{os}^2}{2}\right\} (\cancel{2})(\cancel{\pi})(\cancel{T}) \,\mathrm{sinc}^2\left(\frac{\Omega T}{2\pi}\right) = 0$$

$$= \lim_{T\to\infty} E\left\{\frac{A^2}{2}\right\} \mathrm{sinc}^2(\omega_o T/2\pi) = 0 \quad (\text{It is } 0)$$

$$\Rightarrow \lim_{T\to\infty} \mathrm{Var}\{\hat{\mu}_x^{(T)}\} = 0$$

$\Longrightarrow$ $X(t)$ is ergodic in the mean

Ergodicity in Correlation

$$\hat{R}_{xx}^{(T)}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x^*(t-\tau)\,dt$$

$$\hat{R}_{xx}^{(T)}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} A_0^2 \cos(\omega_0 t + \theta_0)\cos(\omega_0 t - \omega_0\tau + \theta_0)\,dt$$

$$\hat{R}_{xx}^{(T)}(\tau) = \frac{A_0^2}{2T}\int_{-T/2}^{T/2} \left[\cos(2\omega_0 t - \omega_0\tau + 2\theta_0) + \cos(\omega_0\tau)\right]dt$$

$$\hat{R}_{xx}^{(T)}(\tau) = \frac{A_0^2}{2T}\cos(\omega_0\tau)(T) + \frac{A_0^2}{2T}\int_{-T/2}^{T/2}\cos(\cdot)\,dt$$

$$\hat{R}_{xx}^{(T)}(\tau) = \frac{A_0^2}{2}\cos(\omega_0\tau) + \frac{A_0^2}{2T}k(T),$$

$$k(T) = \begin{cases} 0, & T = \frac{2\pi}{2\omega_0}\cdot r,\quad r\in I \\ c(T), & T \neq \frac{2\pi}{2\omega_0}\cdot r,\quad c(T) < T \end{cases}$$

$$E\left\{\hat{R}_{xx}^{(T)}(\tau)\right\} = E\left\{\frac{A_0^2}{2}\right\}\cos(\omega_0\tau) + E\left\{\frac{A_0^2}{2}\right\}\frac{k(T)}{T}$$

$$\lim_{T\to\infty} E\left\{\hat{R}_{xx}^{(T)}(\tau)\right\} = E\left\{\frac{A_0^2}{2}\right\}\cos(\omega_0\tau) = R_{xx}(\tau)$$

On average $\hat{R}_{xx}^{(T)}(\tau) \xrightarrow{T\to\infty} R_{xx}(\tau)$

$$\mathrm{Var}\left\{\hat{R}_{xx}^{(T)}(\tau)\right\} = E\left\{\left[\hat{R}_{xx}^{T}(\tau) - R_{xx}(\tau)\right]^2\right\}$$

$$= E\left\{\left[\frac{A_0^2}{2}\cos(\omega_0\tau) + \frac{A_0^2}{2}\frac{k(T)}{T} - E\left\{\frac{A_0^2}{2}\right\}\cos(\omega_0\tau)\right]^2\right\}$$

$$= E\left\{\left[\left(\frac{A_0^2}{2} - E\left\{\frac{A_0^2}{2}\right\}\right)\cos(\omega_0\tau) + \frac{A_0^2}{2}\frac{k(T)}{T}\right]^2\right\}$$

$$\lim_{T\to\infty} \mathrm{Var}\left\{\hat{R}_{xx}^{(T)}(\tau)\right\} = \left[\sigma_A^2/2\ \cos(\omega_0\tau)\right]^2 \neq 0$$

$\Longrightarrow$ $X(t)$ is not ergodic in correlation

NOTE: $\lim_{T\to\infty} \mathrm{Var}\left\{\hat{R}_{xx}^{(T)}(\tau)\right\} = 0$ only if $\sigma_A^2 = 0$ or A is a constant

This is consistent with the result from the random oscillator example on the web, where A is a constant