Example : Ergodicity

Consider the random process $X(t,\omega)$ defined by the member functions

$\omega = 1$	$x_1(t) = 1$
$\omega = 2$	$x_2(t) = -1$
$\omega = 3$	$x_3(t) = 2$
$\omega = 4$	$x_4(t) = -2$
$\omega = 5$	$x_5(t) = 3$
$\omega = 6$	$x_6(t) = -3$

Assume that knowledge of only one member function is available

$$\hat{\mu}_x \triangleq \frac{1}{T}\int_{-T/2}^{T/2} X(t,\omega_0)\,dt \;, \quad \omega_0 \in \Omega$$

Since the member functions are constant

$\hat{\mu}_x \in \{1,-1,2,-2,3,-3\}$ with values equally likely

$$E\{\hat{\mu}_x\} = \frac{1}{6}\{1+(-1)+2+(-2)+3+(-3)\} = 0$$

Of course this the same result we would have obtained through the ensemble average:

$$\mu_x(t) = E\{X_t(\omega)\} = \frac{1}{6}\{1+(-1)+(2)+(-2)+3+(-3)\}$$

$$= 0$$

$$\Rightarrow \quad E\{\hat{\mu}_x\} = \mu_x$$

$$\Rightarrow \quad \lim_{T\to\infty} E\{\hat{\mu}_x\} = \mu_x \qquad (1)$$

Let us now look at the variance of $\hat{\mu}_x$

$$Var\{\hat{\mu}_x\} = E\{(\hat{\mu}_x - \mu_x)^2\}$$

$$= \frac{1}{6}\{(-1-0)^2 + (1-0)^2 + (2-0)^2 + (-2-0)^2 + (3-0)^2 + (-3-0)^2\} = \frac{28}{6} = \frac{14}{3}$$

$$\Rightarrow \quad \lim_{T\to\infty} Var\{\hat{\mu}_x\} = \frac{14}{3} \neq 0 \qquad (2)$$

Since the variance of the estimate does not go to zero, $\hat{\mu}_x$ is not a consistent estimate

$\Rightarrow$ $X(t, \omega)$ is not ergodic in the mean.