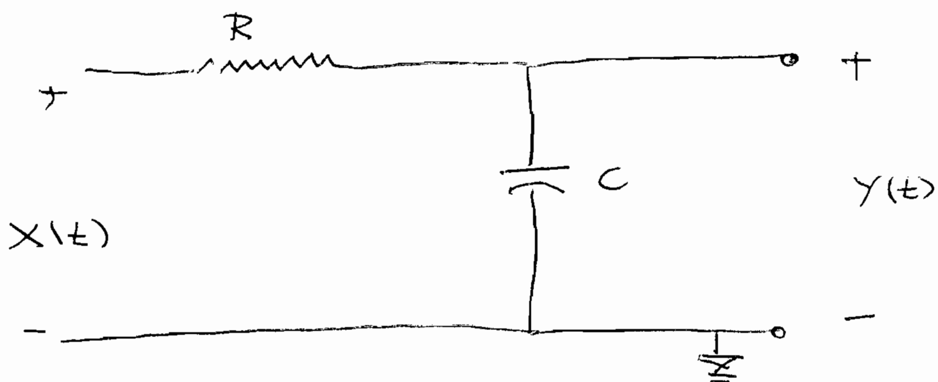
ECE - 541, Fall 2007
Prob. Theory & Stoc. Proc.

Example : LTI processing

Consider a R-C voltage divider circuit with the output tapped across the capacitor. Let $X(t)$, be a zero-mean, Gaussian, white noise input that is applied to the R-C system that is initially at rest.



Let $Y(t)$ be the output response of the R-C circuit to the white-noise input

Lag Domain Calculations:

$$\mu_Y(t) = \mu_X(t) * h(t)$$

In our case, $\mu_X(t) = \mu_X = 0$ (1)

$$\Rightarrow \mu_Y(t) = \mu_Y = 0$$

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau)$$
$$= \sigma_x^2 \, \delta(\tau) * h(\tau)$$
$$= \sigma_x^2 h(\tau)$$

In our case:

$$H(s) = \frac{1/RC}{s + \frac{1}{RC}} , \quad Re\{s\} > -\frac{1}{RC}$$

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\Rightarrow R_{yx}(\tau) = \begin{cases} \frac{\sigma_x^2}{RC} e^{-\tau/RC}, & \tau \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h^*(-\tau)$$
$$= \sigma_x^2 \, (h(\tau) * h^*(-\tau))$$

P.S.D calculations

$$P_{yy}(s) = \sigma_x^2 H(s) H^*(-s^*) = \frac{\sigma_x^2 \left(\frac{1}{RC}\right)^2}{s^2 - \left(\frac{1}{RC}\right)^2} ,$$

$$-\frac{1}{RC} < Re\{s\} < \frac{1}{RC}$$

$$P_{yx}(s) = \sigma_x^2 H(s) = \frac{\sigma_x^2 \frac{1}{RC}}{s + \frac{1}{RC}} , \quad Re\{s\} > -\frac{1}{RC} \quad (3)$$

$$\sigma_y^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_x^2 \, |H(j\Omega)|^2 \, d\Omega$$

$$\sigma_y^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_x^2 \, \frac{a^2}{\Omega^2 + a^2} \, d\Omega \,, \quad \text{where}$$

$$a = \frac{1}{RC}$$

$$\sigma_y^2 = \frac{1}{2\pi} \, \sigma_x^2 \, \frac{a^2}{a} \, \tan^{-1}\left(\frac{\Omega}{a}\right) \Bigg|_{-\infty}^{\infty}$$

$$\sigma_y^2 = \frac{1}{2\pi} \, \sigma_x^2 \, \frac{1}{RC} \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$\sigma_y^2 = \frac{\sigma_x^2}{2RC} \qquad (4)$$

$$R_{yy}(\tau) = \mathcal{F}^{-1}\left\{ \frac{\sigma_x^2 \, a^2}{a^2 + \Omega^2} \right\}$$

$$= \frac{\sigma_x^2 a}{2} \, \mathcal{F}^{-1}\left\{ \frac{2a}{a^2 + \Omega^2} \right\}$$

$$= \frac{\sigma_x^2 a}{2} \, e^{-a|\tau|}$$

$$= \frac{\sigma_x^2}{2RC} \, e^{-|\tau|/RC} \qquad (5)$$

It is in the general case, not possible to evaluate n^{th}-order PDF's of $Y(t)$, however, since $X(t)$ is a G.R.P. $Y(t)$ is also a G.R.P.

$$Y(t) \sim N\left(\mu_Y, \sigma_Y^2\right)$$
$$\sim N\left(0, \frac{\sigma_X^2}{2RC}\right)$$

$$\begin{pmatrix} Y(t_1) \\ Y(t_2) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{\sigma_X^2}{2RC}\begin{pmatrix} 1 & e^{-|\tau|/RC} \\ e^{-|\tau|/RC} & 1 \end{pmatrix}\right)$$

Observations:

(a) $P_{ave}^{X} = \infty$ but $P_{ave}^{Y} = \sigma_Y^2 = \frac{\sigma_X^2}{2RC} < \infty$

or in otherwords filtered WGN is realizable

(b) It is easily verified that

$$\int_{-\infty}^{\infty} |R_{YY}(\tau)|\, d\tau < \infty$$ which implies

$Y(t)$ is ergodic in the general sense

(c) $Y(t)$ is MS continous eventhough $X(t)$ is not.