ECE-541, Fall 2010
Prob. Theory & Stoch. Processes

Example: M.S. Continuity

Let $X(t)$ and $Y(t)$ be two independent WSS processes that are M.S. continuous.

Suppose
$$Z(t) \triangleq X(t) + Y(t)$$
$$W(t) \triangleq X(t)\,Y(t)$$

We know that

$$\lim_{t \to t_0} E\left\{ \left(X(t) - X(t_0) \right)^2 \right\} = 0 \quad , \; t_0 \in \mathbb{R}$$

$$\lim_{t \to t_0} E\left\{ \left(Y(t) - Y(t_0) \right)^2 \right\} = 0 \quad , \; t_0 \in \mathbb{R}$$

Let us now look at the limits:

(a) $\lim\limits_{t \to t_0} E\left\{ \left(Z(t) - Z(t_0) \right)^2 \right\}$ and

(b) $\lim\limits_{t \to t_0} E\left\{ \left(W(t) - W(t_0) \right)^2 \right\}$

For (a):

$$\lim_{t \to t_0} E\left\{ \left(X(t) + Y(t) - X(t_0) - Y(t_0)\right)^2 \right\}$$

$$= \lim_{t \to t_0} E\left\{ \left[(X(t) - X(t_0)) + (Y(t) - Y(t_0)) \right]^2 \right\}$$

$$\leq \lim_{t \to 0} 2\, E\left\{ (X(t) - X(t_0))^2 \right\} + \lim_{t \to t_0} 2\, E\left\{ (Y(t) - Y(t_0))^2 \right\}$$

(||-gram law)

$= 0$ by M.S continuity of $X(t)$ and $Y(t)$

$\Rightarrow$ $Z(t)$ is M.S. continuous.

For (b):

$$\lim_{t \to t_0} E\left\{ \left(X(t)Y(t) - X(t_0)\, Y(t_0) \right)^2 \right\}$$

$$= \lim_{t \to t_0} E\left\{ \left(X(t)Y(t) - X(t_0)\, Y(t) + X(t_0)\, Y(t) - X(t_0)\, Y(t_0) \right)^2 \right\}$$

$$\leq 2 \lim_{t \to t_0} E\left\{ Y^2(t) (X(t) - X(t_0))^2 \right\} + 2 \lim_{t \to t_0} E\left\{ X^2(t_0)\, (Y(t) - Y(t_0))^2 \right\}$$

(|| gram law)

Since $X(t)$ and $Y(t)$ are independent we have

$$E\{Y^2(t)(X(t)-X(t_0))^2\}$$
$$= E(Y^2(t))\,E\{(X(t)-X(t_0))^2\}$$

and

$$E\{X^2(t_0)(Y(t)-Y(t_0))^2\}$$
$$= E\{X^2(t_0)\}\,E\{(Y(t)-Y(t_0))^2\}$$

$$\Longrightarrow \quad \lim_{t\to t_0} E\{(W(t)-W(t_0))^2\}$$
$$\leqq 2\lim_{t\to t_0} E\{Y^2(t)\}\,E\{(X(t)-X(t_0))^2\}$$
$$+\ 2\lim_{t\to t_0} E\{X^2(t_0)\}\,E\{(Y(t)-Y(t_0))^2\}$$

Where we are assuming
$X(t)$ and $Y(t) \in L^2(\Omega)$, $t\in\mathbb{R}$

$\Longrightarrow$ $W(t)$ is M.S. continuous.