ECE-541, Fall 2007
Prob. Theory & Stoch Processes

Example: Prediction

Suppose a zero-mean, WSS random sequence has the following ACF sequence:

$r_{xx}[0] = 1$, $r_{xx}[1] = 0.5$, $r_{xx}[2] = 0.5$,
$r_{xx}[3] = 0.25$

The 3^{rd}-order optimal predictor is:

$$\begin{pmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] \end{pmatrix} \begin{pmatrix} a[1] \\ a[2] \\ a[3] \end{pmatrix} = \begin{pmatrix} r_{xx}[1] \\ r_{xx}[2] \\ r_{xx}[3] \end{pmatrix}$$

Calculating this directly yields

$$\begin{pmatrix} a[1] \\ a[2] \\ a[3] \end{pmatrix} = \begin{pmatrix} 3/8 \\ 3/8 \\ -1/8 \end{pmatrix}$$

Instead let us apply the Levinson-Durbin recursion to the problem:

$$\Gamma_0 = 1, \quad \epsilon_0 = r_{xx}[0] = 1$$

$j = 0$: step

$$\gamma_0 = r_{xx}[j+1] + \sum_{i=1}^{j} a_j[i]\, r_{xx}[j-i+1]$$

$$\gamma_0 = r_{xx}[1] + \cancelto{0}{\sum_{i=1}^{0} a_j[i]\, r_{xx}[j-i+1]}$$

$$\gamma_0 = r_{xx}[1] = 0.5$$

$$\Gamma_1 = -\frac{\gamma_0}{\epsilon_0} = \frac{-r_{xx}[1]}{r_{xx}[0]} = \frac{-0.5}{1} = -0.5$$

$$\underline{a}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$j = 1$: step

$$\gamma_1 = r_{xx}[2] + a_1[1]\, r_{xx}[1]$$

$$= 0.5 - \frac{1}{2}(0.5) = \frac{1}{4}$$

$$\Gamma_2 = -\frac{\gamma_1}{\epsilon_1}, \quad \epsilon_1 = \epsilon_0(1 - \Gamma_1^2) = 1\left(1 - \frac{1}{4}\right) = \frac{3}{4}$$

$$\Gamma_2 = -\frac{\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3}$$

$$\epsilon_2 = \frac{3}{4}\left(1 - \frac{1}{9}\right) = \frac{3}{4} \cdot \frac{8}{9} = \frac{2}{3}$$

$$\underline{a}^{(2)} = \begin{pmatrix} \underline{a}^{(1)} \\ 0 \end{pmatrix} + \Gamma_2 \begin{pmatrix} 0 \\ \underline{a}_R^{(1)} \end{pmatrix}$$

$$\underline{a}^{(2)} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} + -\frac{1}{3} \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$J = 2$: step

$$\gamma_2 = r_{xx}[3] + a_2[1]\, r_{xx}[2] + a_2[2]\, r_{xx}[1]$$

$$\gamma_2 = \frac{1}{4} + \left(-\frac{1}{3}\right)\left(\frac{1}{2}\right) - \frac{1}{3}\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

$$\Gamma_3 = \frac{-\gamma_2}{\epsilon_2} = \frac{+1/12}{2/3} = \frac{1}{8}$$

$$\underline{a}^{(3)} = \begin{pmatrix} a_{(2)} \\ 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 \\ a_2^R \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1/3 \\ -1/3 \\ 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/8 \\ -3/8 \\ 1/8 \end{pmatrix}$$

- Note that the coefficients $\frac{3}{8}, \frac{3}{8}, -\frac{1}{8}$ obtained by direct inversion, appear in sign reversal

- Note that the sequence $\left[1, \frac{3}{4}, \frac{2}{3}, \frac{21}{32}\right]$ is a decreasing sequence of prediction variances