ECE - 541, Fall 2007
Stochastic Processes
&
Probability Theory

Example: σ- algebras

Ω : Toss of a fair coin
$= \{H, T\}$

Subsets of Ω :

$$\{H\}$$
$$\{T\}$$
$$\{H\} \cup \{T\} = \Omega$$
$$\{H\} \cap \{T\} = \phi$$

Power Set : $2^{\Omega} = \{\{H\}, \{T\}, \Omega, \phi\}$

$$\sigma(\Omega) = \{\{H\}, \{T\}, \Omega, \phi\}$$

This is also the only possible σ- algebra because Ω is finite. In this case the measure allocated to $\{H\}$ & $\{T\}$ are :

$$\mu(\{H\}) = \frac{1}{2} = P_r\{\{H\}\}$$
$$\mu(\{T\}) = \frac{1}{2} = P_r\{\{T\}\}$$

Now suppose we consider the random experiment of throwing a dart on the $\begin{pmatrix}\text{unit}\\\text{disc}\end{pmatrix}$ described by

$$D \triangleq \{(x,y) \ni x^2+y^2 \leq 1\}$$

The outcome is considered a "miss" if the dart lands outside of D, i.e.,

$$\text{"miss"} \triangleq \{(x,y) \ni x^2+y^2 > 1\}$$

The outcome is considered a "bulls-eye" if it lands right in the center

$$\text{"bulls-eye"} \triangleq \{(0,0)\}$$

Here one possible formulation for $\sigma(\Omega)$ would be:

$$\sigma_1(\Omega) = \{\phi, D, \text{"miss"}, \Omega\}$$ and this would correspond to the minimal σ-algebra defined on Ω

Let $B \cap D$ represent the Borel subsets of D, i.e, the set of all possible open sets contained in D along with their union and complement

Another possible σ-algebra for the dart experiment is :

$$\sigma_2(\Omega) = \{\phi, B \cap D, \text{'miss'}, \Omega\}$$

This σ-algebra is however, much larger than σ_1 since it is not minimal.

Suppose the space under consideration is the real line $\mathbb{R}$, then the corresponding σ-field is a special field called the Borel-field on $\mathbb{R}$ and is composed of the set of all open intervals of the form (a, b), with their complements, unions, and intersections. This is needed so that a measure can be associated with them

One important measure that will be discussed in the class is of the form

$\mu_L((a, b)) = b - a$, and is often termed as the Lebesgue measure
The sets $A \in B$, where μ is defined are called Lebesgue measurable