

## Example: Optimal Smoothing

Consider a smoothing example, where the observations are just a noise contaminated version of the SOI:  $y(t) = x(t) + v(t)$ , where  $v(t)$  is zero-mean, unit variance spectrally white observation noise. Specifically the power spectral density of the SOI,  $x(t)$ , is of the form:

$$P_{xx}(\Omega) = \frac{2\alpha}{\alpha^2 + \Omega^2}, \quad \Omega \in \mathbf{R}.$$

This power-spectrum can be factorized using analytic continuation as:

$$P_{xx}(s) = \frac{2}{\alpha} \left( \frac{\alpha}{\alpha + s} \right) \left( \frac{\alpha}{\alpha - s} \right), \quad |\sigma| < \alpha.$$

The power-spectrum of the observation process is given by:

$$P_{yy}(s) = 1 + \frac{2\alpha}{\alpha^2 - s^2} = \frac{2\alpha + \alpha^2 - s^2}{\alpha^2 - s^2}.$$

This power-spectrum can be factorized via:

$$P_{yy}(s) = \left( \frac{s - \beta}{s - \alpha} \right) \left( \frac{s + \beta}{s + \alpha} \right),$$

where  $\beta = \sqrt{2\alpha + \alpha^2}$ . The cross power-spectrum  $P_{xy}(s)$  is given by:

$$P_{xy}(s) = P_{xx}(s) = \frac{2}{\alpha} \left( \frac{\alpha}{\alpha + s} \right) \left( \frac{\alpha}{\alpha - s} \right).$$

The optimal Wiener smoother is then given by:

$$H_{\text{opt}}(s) = \frac{P_{xy}(s)}{P_{yy}(s)} = \frac{P_{xx}(s)}{P_{yy}(s)} = \frac{2\alpha}{2\alpha + \alpha^2 - s^2}.$$

The optimal causal Wiener smoother is given by:

$$\begin{aligned} H_{\text{cau}}(s) &= \frac{1}{P_{yy}^+(s)} \left( \frac{P_{xx}(s)}{P_{yy}^-(s)} \right)_+ = \left( \frac{s + \alpha}{s + \beta} \right) \left[ \left( \frac{2\alpha}{\alpha^2 - s^2} \right) / \left( \frac{s - \beta}{s - \alpha} \right) \right]_+ \\ &= \left( \frac{2\alpha}{\alpha + \beta} \right) \frac{1}{s + \beta} \end{aligned}$$

We may then obtain the MMSE in each case as:

$$\begin{aligned} \text{MMSE}_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{vv}(\Omega) H_{\text{opt}}(\Omega) d\Omega = \frac{\alpha}{\beta} \\ \text{MMSE}_2 &= R_{xx}(0) - \int_0^{\infty} h_{\text{opt}}(\tau) R_{xx}(\tau) d\tau = \frac{2\alpha}{\alpha + \beta}. \end{aligned}$$