Sufficiency Of Second–Order Statistics

Consider a fourth-order Gaussian random vector $X = [X_1, X_2, X_3, X_4]^T$, where the components are zero-mean Gaussian random variables with a associated covariance matrix $\mathbf{C_{xx}}$. Our goal in this exercise is to prove that if a random vector X has Gaussian statistics then second-order statistics are sufficient for completely characterizing the vector.

The first relevant result is the n-th order PDF of the Gaussian random vector:

$$f_X(x) = K(\mathbf{C}_{xx}) \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}_{xx}^{-1}(\mathbf{x}-\mathbf{m})\right)$$

that is completely parameterized by its mean vector $\mathbf{m}$ and covariance matrix $\mathbf{C}_{xx}$. Consequently any n-th order statistic derived from this PDF is going to depend only on the doublet $(\mathbf{m}, \mathbf{C}_{xx})$. The second relevant result for proving our claim is the characteristic function of the Gaussian random vector:

$$\Psi_x(v_1, v_2, v_3, v_4) = \exp\left(j\mathbf{m}^T\mathbf{v} - \frac{1}{2}\mathbf{v}^T\mathbf{C}_{xx}\mathbf{v}\right).$$

Towards this purpose, let us first evaluate the fourth-order cummulant of the random vector X as:

$$E\{X_1X_2X_3X_4\} = \frac{1}{j^4}\frac{\partial^4\Psi_x(v_1, v_2, v_3, v_4)}{\partial v_1\partial v_2\partial v_3\partial v_4}, \quad v_1 = v_2 = v_3 = v_4 = 0.$$

Setting $\mathbf{m} = \mathbf{0}$ and expanding the exponent in the characteristic function with a Taylor series yields:

$$\Psi_x(j\mathbf{v}) = 1 - \frac{1}{2}\mathbf{v}^T\mathbf{C}_{xx}\mathbf{v} + \frac{1}{8}\left(\mathbf{v}^T\mathbf{C}_{xx}\mathbf{v}\right)^2 + \quad H.O.T.$$

Note that the only term in this expansion that will remain after the four–time partial derivative operation and the DC boundary condition, i.e., $\mathbf{v} = \mathbf{0}$ is the third term. Expanding just this third term we can write:

$$\frac{1}{8}\left(\mathbf{v}^T\mathbf{C}_{xx}\mathbf{v}\right)^2 = \frac{1}{8}\sum_{p=1}^{4}\sum_{q=1}^{4}\sum_{r=1}^{4}\sum_{s=1}^{4} v_p\sigma_{pq}v_qv_r\sigma_{rs}v_s.$$

Upon four–time differentiation and application of the DC boundary condition the only non zero terms remaining will be the terms:

$$\sum_{p,q,r,s,\ p\neq q\neq r\neq s}^{4} \frac{1}{8}\sigma_{pq}\sigma_{rs}$$

Noting further that the covariance function is symmetric and that there are eight possible repetitions for the three distinct terms in the sum we have:

$$E\{X_1X_2X_3X_4\} = \sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}.$$

This is a fundamental result regarding the statistics of the Gaussian random vector in that the fourth-order moment is just the sum of products of second-order moments. Consequently no other information other than second-order statistics is needed to evaluate higher-order moments. An immediate consequence of this theorem is the result that:

$$E\{X^4\} = 3\sigma^4 \iff \frac{E\{X^4\}}{\sigma^4} = 3.$$

which states that the Kurtosis of a Gaussian source is 3. The other fundamental result that can obtained from this analysis is one that relates to the third-order moment $E\{X_1X_2X_3\}$. Since none of the terms in the expansion for the characteristic function will contain triplets of the form $\gamma_1\gamma_2\gamma_3$ that will survive a three–time partial differentiation and DC conditions we have:

$$E\{X_1X_2X_3\} = 0$$

In other words, the information in the odd-order moments of the Gaussian random vector is zero and the information in the even-order moments can be obtained from second-order moments.