Impulse Sampling of Random Signals

Consider a WSS, zero mean random signal $x_c(t)$ that has a PSD $P_{x_c x_c}(\omega)$. This signal is sampled at a rate $f_s = \frac{1}{T_s}$

$$R_{xx}[k] = E\left\{x[n]\,x[n-k]\right\}$$

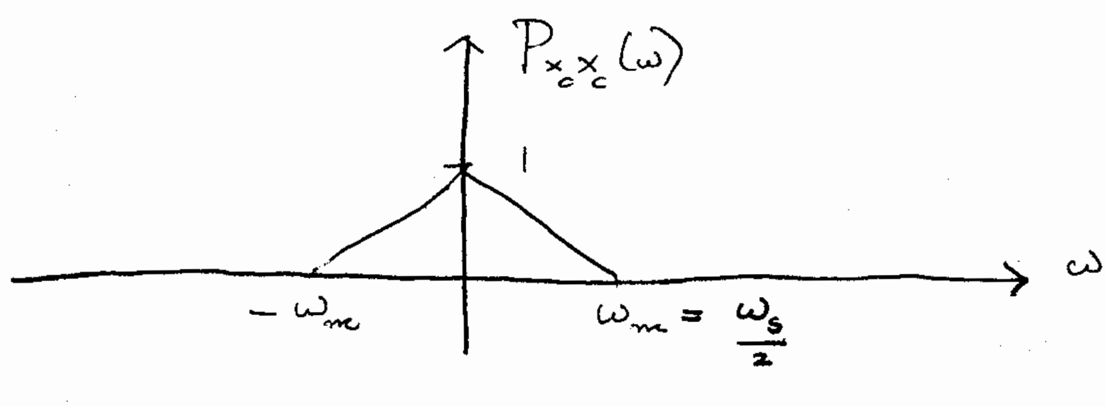
$$R_{xx}[k] = E\left\{x_c(nT_s)\,x_c((n-k)T_s)\right\}$$

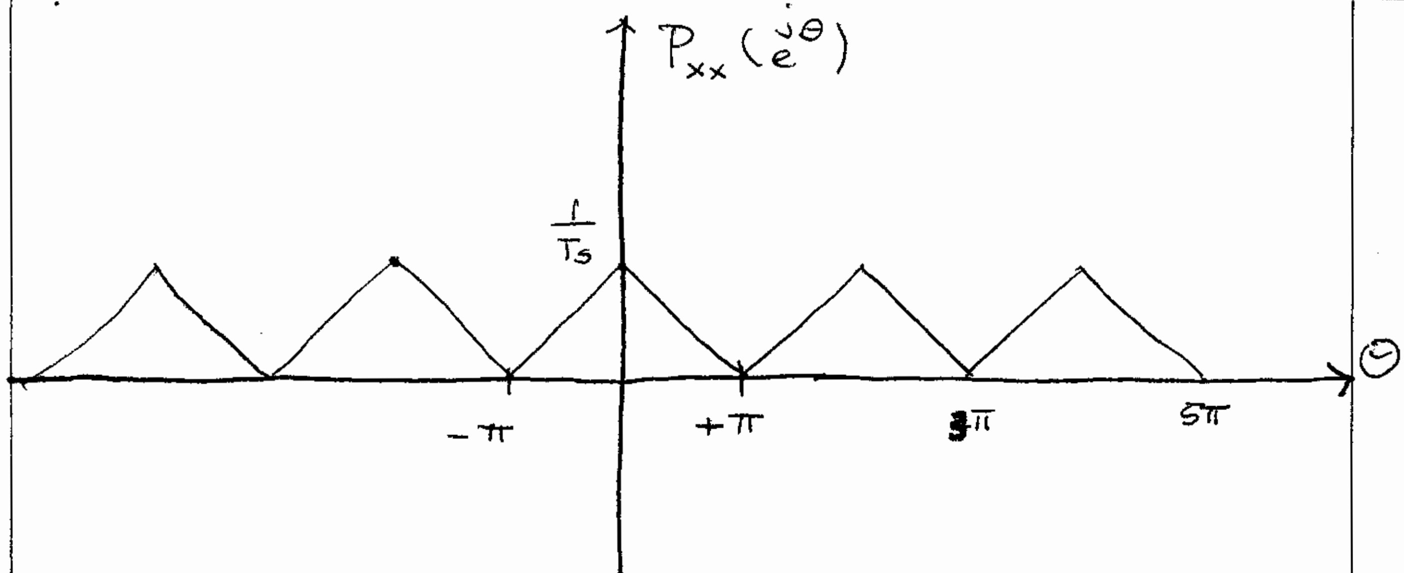
$$R_{xx}[k] = R_{x_c x_c}(kT_s) = R_{x_c x_c}(\tau)\Big|_{\tau = kT_s}$$

$$P_{xx}(z)\Big|_{z = e^{j\theta}} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} P_{x_c x_c}\left(\frac{\theta - 2k\pi}{T_s}\right)$$

(PSD Alias Sum)

The spectral copies $P_{x_c x_c}\left(\frac{\theta - 2k\pi}{T_s}\right)$ will not overlap if $\omega_s \geq 2\omega_m$, where ω_m is the maximum PSD content





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$$X_s(t) = X_c(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$X_s(t) = X_c(t) \left(\frac{1}{T_s}\right) \sum_{k=-\infty}^{\infty} \exp\left(jk \frac{2\pi}{T_s} t\right)$$

$$R_{x_c x_c}(\tau) = E\{X_c(t) X_c(t-\tau)\}$$

$$R_{x_s x_s}(t,\tau) = E\{X_s(t) X_s(t-\tau)\}$$

$$= E\left\{ \left(\frac{1}{T_s}\right)^2 \sum_{m,n} \exp\left(j \frac{2\pi}{T_s} (m+n) t\right) \exp\left(-j \frac{2\pi}{T_s} n \tau\right) X_c(t) X_c(t-\tau) \right\}$$

$$R_{x_s x_s}(t,\tau) = \left(\frac{1}{T_s}\right)^2 R_{x_c x_c}(\tau) \sum_{m} \sum_{n=-\infty}^{\infty} \exp\left(j \frac{2\pi}{T_s} (m+n) t\right) \exp\left(-j \frac{2\pi}{T_s} n \tau\right)$$

$X_s(t)$ is not stationary but cyclostationary with parameter T_s

Time - Averaged ACF:

$$\tilde{R}_{x_s x_s}(\tau) = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} R_{x_s x_s}(t, \tau) \, dt$$

$$\tilde{R}_{x_s x_s}(\tau) = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \left(\frac{1}{T_s}\right)^2 R_{x_c x_c}(\tau) \sum_m \sum_n \exp\left(j \frac{2\pi}{T_s} (m+n) t\right) \exp\left(-j \frac{2\pi}{T_s} n \tau\right) \, dt$$

$$\tilde{R}_{x_s x_s}(\tau) = \left(\frac{1}{T_s}\right)^2 R_{x_c x_c}(\tau) \sum_{n=-\infty}^{\infty} \exp\left(-j \frac{2\pi}{T_s} n \tau\right) \cdot \sum_{m=-\infty}^{\infty} \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \exp\left(j \frac{2\pi}{T_s} (m+n) t\right) dt$$

$$\tilde{R}_{x_s x_s}(\tau) = \left(\frac{1}{T_s}\right)^2 R_{x_c x_c}(\tau) \sum_{n=-\infty} \sum_{m=-\infty}^{\infty} \exp(-j \omega_s n \tau) \, \delta[m+n]$$

$$\tilde{R}_{x_s x_s}(\tau) = \frac{1}{T_s} R_{x_c x_c}(\tau) \left[\left(\frac{1}{T_s}\right) \sum_{n=-\infty}^{\infty} \exp(j \omega_s n \tau)\right]$$

$$\tilde{R}_{x_s x_s}(\tau) = \frac{R_{x_c x_c}(\tau)}{T_s} \sum_{k=-\infty}^{\infty} \delta(t - k T_s)$$

$$\tilde{P}_{x_s x_s}(\omega) = \left(\frac{1}{T_s}\right)^2 \sum_{k=-\infty}^{\infty} P_{x_c x_c}\left(\omega - k \frac{2\pi}{T_s}\right)$$

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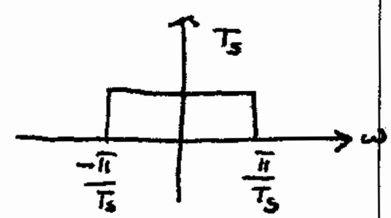
Reconstruction:

$$X_s(t) = \sum_{n=-\infty}^{\infty} X_c(nT_s)\, \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT_s)$$



$$\hat{X}_c(t) = X_s(t) * h_{LP}(t),$$

$$h_{LP}(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right) = \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\left(\frac{\pi t}{T_s}\right)}$$

$$\hat{X}_c(t) = \sum_{n=-\infty}^{\infty} x[n]\, \frac{\sin\left(\frac{t}{T_s} - n\pi\right)}{\left(\frac{t}{T_s} - n\pi\right)}$$

$$R_{x_c x_c}(\tau) = \sum_{m=-\infty}^{\infty} R_{xx}[m]\, \operatorname{sinc}\left(\frac{t}{T_s} - m\right)$$

(Sinc - Interpolation)

Example:

$$R_{x_c x_c}(\tau) = \exp\left(-a|\tau|\right)$$

$$P_{x_c x_c}(\omega) = \frac{2a}{a^2+\omega^2}$$

This is not a bandlimited PSD

Impulse - sampling will cause Aliasing

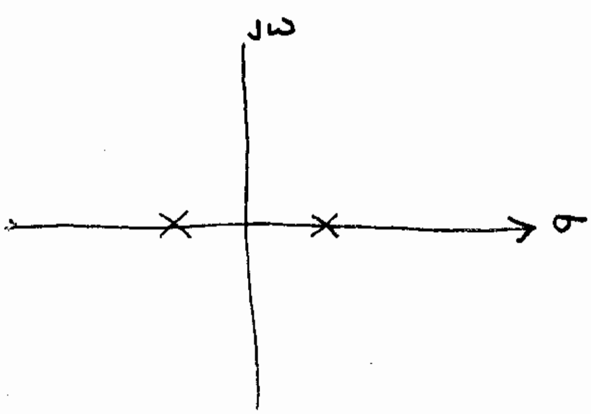
$$R_{xx}[k] = \exp(-aT_s|k|)$$

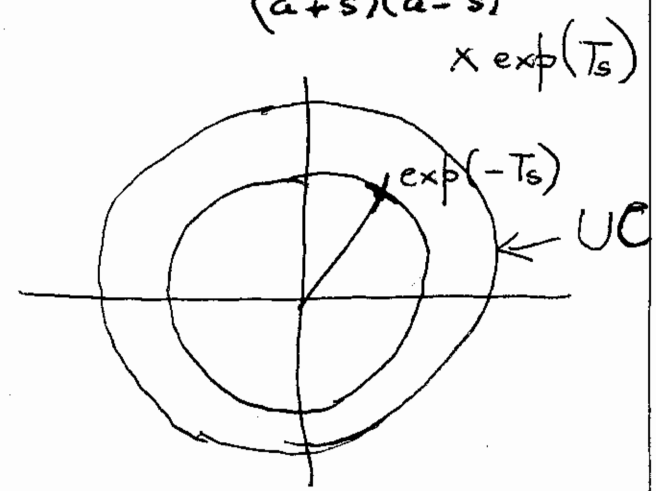
$$P_{xx}(e^{j\theta}) = \frac{1 - \exp(-2aT_s)}{1 + \exp(-2aT_s) - 2\exp(-T_s)\cos\theta}$$

$$P_{xx}(z) = \frac{1-\exp(-2aT_s)}{\left[1 + \exp(-T_s)z^{-1}\right]\left[1-\exp(-T_s)z\right]}$$

In contrast

$$P_{x_c x_c}(s) = \frac{2a}{a^2-s^2} = \frac{2a}{(a+s)(a-s)}$$





$$P_{ave}^{x_c} = R_{x_c x_c}(0) = 1$$

$$P_{ave}^{x} = R_{xx}[0] = 1$$

$$R_{x_s x_s}(\tau) = \frac{1}{T_s} R_{x_c x_c}(\tau) \sum_{n=-\infty}^{\infty} \delta(\tau - nT_s)$$

$$R_{x_s x_s}(\tau) = \left(\frac{1}{T_s}\right) \sum_{n=-\infty}^{\infty} \exp(-aT_s|n|)\, \delta(\tau - nT_s)$$

$$F\left\{R_{x_s x_s}(\tau)\right\} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \exp(-aT_s|n|) \exp(-j\omega n T_s)$$

$$= \frac{1}{T_s} \, \mathrm{DTFT}\left\{\exp(-aT_s|n|)\right\}\Big|_{\theta = \omega T_s}$$

$$= \frac{1}{T_s} \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos(\omega T_s)}, \quad |\omega| \le \pi/T_s$$

$$P_{x_s x_s}(\omega) = \left(\frac{1}{T_s}\right)\left[\frac{(1-\alpha^2)}{1+\alpha^2 - 2\alpha\cos(\omega)}\right], \quad |\omega| \le \frac{\pi}{T_s}$$

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PAM (Digital Modulation)

$$X(t) = \sum_{k=-\infty}^{\infty} a[k]\, p(t - kT_b)$$

$a[k]$: Discrete Symbols (White)

T_b : Bit period or Symbol period

$p(t)$: Pulse Shaping function

$$X(t) = \left[\sum_{k=-\infty}^{\infty} a[k]\, \delta(t - kT_b)\right] * p(t)$$

$p(t)$ serves the role of the impulse response of the interpolating LPF and the quantity in the parenthesis is the signal $a_s(t)$ [Quasi - discrete, quasi - continuous Signal]

From my previous class notes PAM signals are cyclostationary with parameter T_b

$$\tilde{R}_{ss}(\tau) = \frac{\sigma_a^2}{T_b}\, p(\tau) * p^*(-\tau)$$

$$\tilde{P}_{ss}(\omega) = \frac{\sigma_a^2}{T_b}\, |P(\omega)|^2$$