Example: Transmission through LTI systems

$$x[n] \longrightarrow \boxed{\dfrac{1}{1-az^{-1}}} \longrightarrow y[n]$$

$|z| > a$

zero mean
WSS
White noise

Input - output Difference Eq:

$$y[n] = a\,y[n-1] + x[n]$$

Recursion is run over $-\infty \leq n \leq \infty$

$$h[k] = a^k u[k]$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-ae^{-j\omega}} = H(e^{j\omega})$$

$$|H(e^{j\omega})|^2 = \frac{1}{1-2a\cos\omega + a^2}, \quad |\omega| \leq \pi$$

Time - domain analysis:

$$\begin{aligned} r_{yx}[k] &= E\{y[n]\,x^*[n-k]\} \\ &= E\{(a\,y[n-1] + x[n])(x^*[n-k])\} \\ &= a\,r_{yx}[k-1] + r_{xx}[k] \qquad (1) \end{aligned}$$

If $P_{xx}(e^{j\omega}) = \sigma_x^2 \; , \; |\omega| \leq \pi$

$$r_{xx}[k] = \sigma_x^2 \, \delta[k]$$

$$r_{yx}[k] = a \, r_{yx}[k-1] + \sigma_x^2 \, \delta[k]$$

$$r_{yx}[k] = a \, r_{yx}[k-1] \; , \quad k > 0$$

$$r_{yx}[k] = a^k \, r_{yx}[0] \, u[k]$$

$$r_{yx}[0] = a \, r_{yx}[-1] + \sigma_x^2 = \sigma_x^2$$

since $r_{yx}[-1] = E\{y[n] \, x[n+1]\} = 0$

$$r_{yx}[k] = a^k \, \sigma_x^2 \, u[k] \qquad (2)$$

Alternatively we could have obtained this via:

$$\begin{aligned} r_{yx}[k] &= r_{xx}[k] * h[k] \\ &= (\sigma_x^2 \, \delta[k]) * (a^k \, u[k]) \\ &= \sigma_x^2 \, a^k \, u[k] \end{aligned}$$

$$\begin{aligned} r_{yy}[k] &= E\{y[n] \, y^*[n-k]\} \\ &= E\{(a \, y[n-1] + x[n])(y^*[n-k])\} \end{aligned}$$

$$r_{yy}[k] = a \, r_{yy}[k-1] + r_{xy}[k] \qquad (3)$$

$$\begin{aligned} r_{xy}[k] &= E\{y^*[n-k] \, x[n]\} = \left[E\{y[n-k] \, x^*[n]\}\right]^* \\ &= r_{yx}^*[-k] \end{aligned}$$

For real $x[n]$ & $y[n]$: $r_{xy}[k] = r_{yx}[-k]$

(4)

$$r_{yy}[k] = a\, r_{yy}[k-1] + r_{yx}[-k]$$
$$r_{yy}[k] = a\, r_{yy}[k-1] + \sigma_x^2 h[-k] \quad (5)$$

Combining (1) & (5) we obtain the Yule-Walker recursion

For a causal $h[k]$, $h[k] = 0$, $k < 0$

$$r_{yy}[k] = a\, r_{yy}[k-1] + \sigma_x^2 (0), \quad k > 0$$
$$\Rightarrow r_{yy}[k] = a^k r_{yy}[0], \quad k > 0 \quad (6)$$

$$r_{yy}[0] = a\, r_{yy}[-1] + \sigma_x^2 h[0]$$
$$r_{yy}[0] = a\, r_{yy}[1] + \sigma_x^2 h[0] \quad (7)$$

$$r_{yy}[k],\ k < 0 = r_{yy}[-k],\ k > 0$$
$$= a^{-k} r_{yy}[0] \quad (8)$$

Combining (6), (7), (8)

$$r_{yy}[k] = \begin{cases} a^k\, r_{yy}[0], & k > 0 \\ a^{-k}\, r_{yy}[0], & k < 0 \\ \dfrac{\sigma_x^2 h[0]}{1-a^2}, & k = 0 \end{cases}$$

$$\boxed{r_{yy}[k] = \frac{\sigma_x^2}{1-a^2} h[0]\, a^{+|k|}}$$

Frequency Domain Analysis

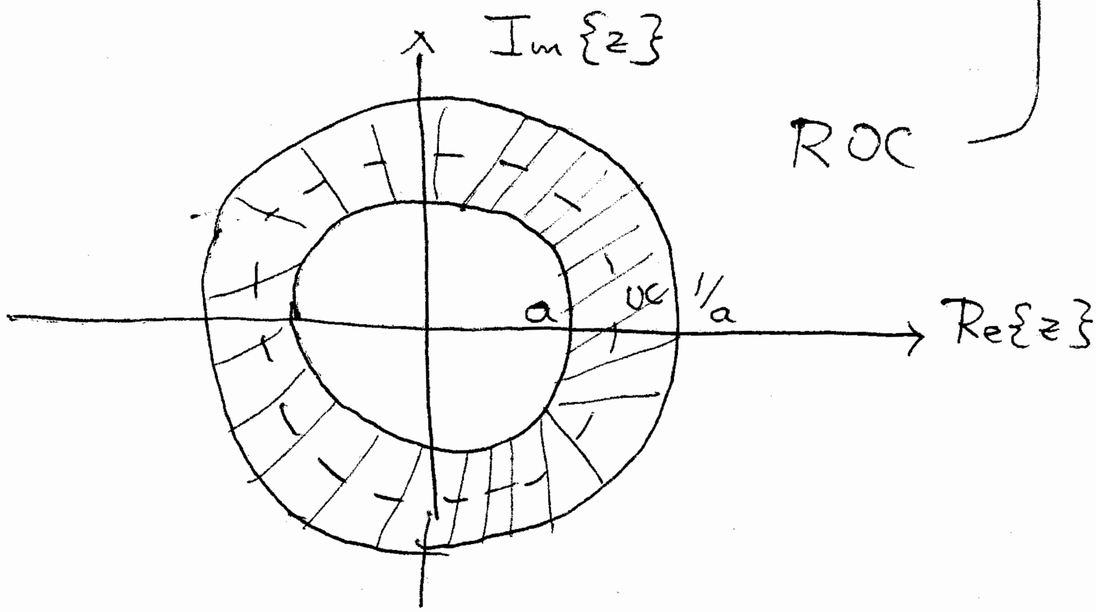
$$P_{yx}(e^{j\omega}) = H(e^{j\omega})\,P_{xx}(e^{j\omega})$$
$$= \frac{\sigma_x^2}{1 - a e^{-j\omega}}, \quad |\omega| \le \pi$$

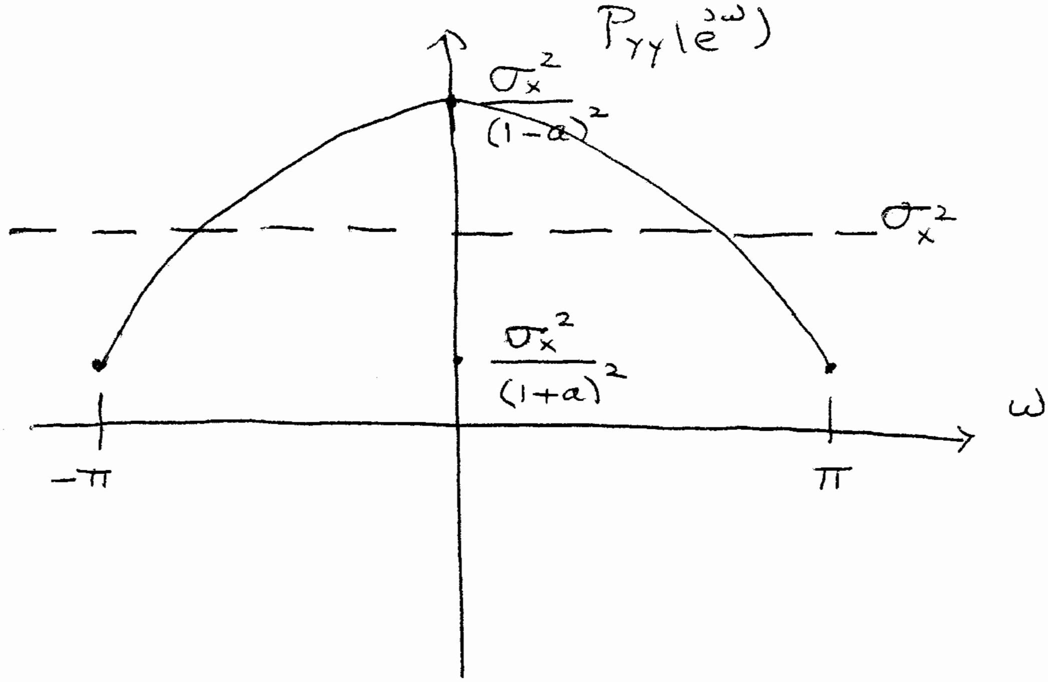
$$r_{yx}[k] = \sigma_x^2\, a^k\, u[k] \quad (\text{IDTFT})$$

$$P_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\,\sigma_x^2$$
$$= \frac{\sigma_x^2}{1 + a^2 - 2a\cos\omega}, \quad |\omega| \le \pi$$

$$P_{yy}(z) = P_{xx}(z)\,H(z)\,H^*(1/z^*)$$
$$= \frac{\sigma_x^2}{(1 - a z^{-1})(1 - a z)}, \quad a < |z| < 1/a$$

ROC





$$r_{YY}[k] = \mathcal{F}^{-1}\{P_{YY}(e^{j\omega})\}$$

$$= \left(\frac{\sigma_x^2}{1-a^2}\right) \mathcal{F}^{-1}\left\{\frac{1-a^2}{1+a^2-2a\cos\omega}\right\}$$

$$r_{YY}[k] = \frac{\sigma_x^2}{1-a^2} \; a^{|k|}$$

$$a^{|k|} \rightleftharpoons \frac{1-a^2}{(1-az^{-1})(1-az)}, \quad a < |z| < \frac{1}{a}$$

$$a^{|k|} \rightleftharpoons \frac{1-a^2}{1+a^2-2a\cos\omega}, \quad |\omega| \leq \pi$$

$$P_{ave}^{Y} = \frac{\sigma_x^2}{1-a^2} < \infty \quad \text{if } |a| \neq 1$$