ECE-541, Fall 2010

Prob. Theory & Stochastic Processes

Example : M.S. Convergence

Consider a sequence of R.V's defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that converge in probability to the R.V $X(\omega)$:

$$\lim_{n\to\infty} \Pr\{ |X_n(\omega) - X(\omega)| > \epsilon \} = 0, \quad \text{for any } \epsilon > 0$$

Suppose the PDF of $X_n(\omega)$ is $\ni$:

$$f_{X_n}(x) = 0 , \quad |x| > x_0 , \quad n > N$$

Since $X_n(\omega) \xrightarrow{\mathcal{D}} X(\omega)$, we have :

$$f_X(x) = 0 , \quad |x| > x_0 \quad (\text{compact PDF})$$

Define $|X_n(\omega) - X(\omega)| \triangleq Z_n(\omega)$

$$\Pr\{ |X_n(\omega) - X(\omega)| > 2x_0 \}$$

$$\leq \Pr\{ |X_n(\omega)| > x_0 \} + \Pr\{ |X(\omega)| > x_0 \}$$

$$0 \leq \lim_{n\to\infty} \Pr\{ Z_n(\omega) > 2x_0 \} \leq \lim_{n\to\infty} \cancelto{0}{\Pr\{ |X_n(\omega)| > x_0 \}}$$

$$+ \cancelto{0}{\Pr\{ |X(\omega)| > x_0 \}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \Pr\{ Z_n(\omega) > 2x_0 \} = 0$$

Let us now evaluate:

$$0 \le E\{[X_n(\omega) - X(\omega)]^2\} = E\{Z_n^2(\omega)\}$$

$$E\{Z_n^2(\omega)\} \triangleq \int_{-\infty}^{\infty} z^2 f_{Z_n}(z)\,dz = \int_{0}^{\infty} z^2 f_{Z_n}(z)\,dz$$

$$E\{Z_n^2(\omega)\} = \int_{0}^{\epsilon} z^2 f_{Z_n}(z)\,dz + \int_{\epsilon}^{2x_0} z^2 f_{Z_n}(z)\,dz$$

$$0 \le E\{Z_n^2(\omega)\} \le \epsilon^2 \int_{0}^{\epsilon} f_{Z_n}(z)\,dz + 4x_0^2 \int_{\epsilon}^{2x_0} f_{Z_n}(z)\,dz.$$

$$\le \epsilon^2 + 4x_0^2 \Pr\{Z_n > \epsilon\}$$

First applying limit as $n \to \infty$

$$\lim_{n\to\infty} E\{Z_n^2(\omega)\} \le \epsilon^2 + 4x_0^2 \overbrace{\lim_{n\to\infty} \Pr\{Z_n > \epsilon\}}^{\to 0}$$

$$= \epsilon^2$$

Next apply limits as $\epsilon \to 0_+$

$$0 \le \lim_{\substack{n\to\infty \\ \epsilon\to 0_+}} E\{Z_n^2(\omega)\} \le \lim_{\substack{n\to\infty \\ \epsilon\to 0_+}} \epsilon^2 = 0$$

$$\Rightarrow \quad X_n(\omega) \xrightarrow[m.s.]{n\to\infty} X(\omega)$$

→ If $X_n(\omega) \xrightarrow[n\to\infty]{l.p.} X(\omega)$ and $X_n(\omega)$ and $X(\omega)$ have a compact PDF then $X_n(\omega) \xrightarrow{m.s} X(\omega)$