Markov Processes

$$f\left(X(t_n)\,\middle|\,_{X(t_{n-1})=x_{n-1},\ \cdots\cdots\ X(t_1)=x_1}\right)$$

$$= f\left(X(t_n)\,\middle|\,_{X(t_{n-1})=x_{n-1}}\right) \quad \text{— (1)}$$

$$f\left(X(t_n)\,\middle|\,_{X(t),\quad t \le t_{n-1}}\right)$$

$$= f\left(X(t_n)\,\middle|\,_{X(t_{n-1})=x_{n-1}}\right) \quad \text{— (2)},$$

where $\{t_i\}_{i=1}^{n}$, $t_1 < t_2 < t_3, \ldots < t_n \in T$

Both these definitions are

$\Rightarrow$ equivalent

$\Rightarrow$ Markov process retains memory of only the most recent past.

LHS of (1)

$$f\left(x(t_n)\Big|_{x(t_{n-2})=x_{n-2}}\right) = f\left(x(t_n)\Big|_{x(t_{n-1})=x_{n-1}}\right)$$

$$f\left(x(t_n)\Big|_{x(t_{n-3})=x_{n-3}}\right) = f\left(x(t_n)\Big|_{x(t_{n-1})=x_{n-1}}\right)$$

$$\vdots$$

$$f\left(x(t_n)\Big|_{x(t_1)=x_1}\right) = f\left(x(t_n)\Big|_{x(t_{n-1})=x_{n-1}}\right)$$

Where $\{t_i\}_{i=1}^{n} \in T$ & $t_1 < t_2, \ldots < t_n$

$$\Rightarrow f\left(x(t_n)\Big|_{x(t_{n-1})=x_{n-1},\ x(t_{n-2})=x_{n-2},\ \ldots\ldots}\right)$$
$$= f\left(x(t_n)\Big|_{x(t_{n-1})=x_{n-1}}\right)$$

$\Rightarrow$ (1) & (2) are equivalent definitions.

Chain Rule:

$$f(x_1, x_2, \dots x_n) = f\left(x_n / x_{n-1}, \dots x_1\right) \cdot f\left(x_{n-1} / x_{n-2}, \dots x_1\right) \dots\dots f\left(x_2 / x_1\right) f(x_1)$$

Using Markov Property

$$f\left(x_n / x_{n-1}, \dots x_1\right) = f\left(x_n / x_{n-1}\right)$$

$$f\left(x_{n-1} / x_{n-2}, \dots x_1\right) = f\left(x_{n-1} / x_{n-2}\right)$$

Chain Rule for Markov Processes

$$f(x_1, \dots, x_n) = f\left(x_n / x_{n-1}\right) f\left(x_{n-1} / x_{n-2}\right) \dots f\left(x_2 / x_1\right) f(x_1)$$

Markov Process is a Martingale ?

$$E\left\{x_n \middle| x_{n-1}, \ldots x_1\right\}$$

$$= \int_{-\infty}^{\infty} x_n \, f\left(X(t_n) \middle|_{X(t_{n-1})=x_{n-1}, \ldots X(t_1)=x_1}\right) dx_n$$

$$= \int_{-\infty}^{\infty} x_n \, f\left(X(t_n) \middle|_{X(t_{n-1})=x_{n-1}}\right) dx_n$$

$$= E\left\{X(t_n) \middle| X(t_{n-1}) = x_{n-1}\right\} \stackrel{?}{=} x_{n-1}$$

$\Rightarrow$ Markov Process is a martingale. if the process has a constant mean

$\Rightarrow$ In general Markov process is not a Martingale.

Markov Process is a Markov process even if time is reversed

$$f\left(X(t_n)\,\Big|\,X(t_{n+1})=x_{n+1}, \ldots\ldots X(t_{n+k})=x_{n+k}\right)$$

$$= f\left(X(t_n)\,\Big|\,X(t_{n+1})=x_{n+1}\right)$$

<u>Proof</u>

$$f\left(X(t_n)\,\Big|\,X(t_{n+1})=x_{n+1}, \ldots\ldots x(t_{n+k})=x_{n+k}\right)$$

$$= \frac{f(x_n, x_{n+1} \ldots\ldots x_{n+k})}{f(x_{n+1}, x_{n+2} \ldots\ldots x_{n+k})}$$

$$= \frac{\cancel{f(x_{n+k}/x_{n+k-1})}\,\cancel{f(x_{n+k-1}/x_{n+k-2})}\cancel{\ldots}\, f(x_{n+1}/x_n)\, f(x_n)}{\cancel{f(x_{n+k}/x_{n+k-1})}\,\cancel{f(x_{n+k-1}/x_{n+k-2})} \ldots\ldots \cancel{f(x_{n+2}/x_{n+1})} \cdot f(x_{n+1})}$$

$$= \frac{f(x_{n+1}/x_n)\, f(x_n)}{f(x_{n+1})} = \frac{f(x_n, x_{n+1})}{f(x_{n+1})}$$

$$= f\left(X(t_n)\,\Big|\,x(t_{n+1}) = x_{n+1}\right)$$

If the present is specified then the past is independent of the future in the sense that if $k < m < n$

$$f(x_n, x_k \mid x_m) = f(x_n/x_m)\, f(x_k/x_m)$$

Proof:

$$\text{LHS} = f(x_n, x_k / x_m) \quad \text{for } k < m < n$$

$$= \frac{f(x_n, x_k, x_m)}{f(x_m)}$$

$$= \frac{f(x_n/x_m)\, f(x_m/x_k)\, f(x_k)}{f(x_m)}$$

$$= f(x_n/x_m)\, \frac{f(x_k, x_m)}{f(x_m)}$$

$$= f(x_n/x_m)\, f(x_k/x_m)$$

$$= \text{RHS}$$

Conditional Transitional Densities

$$f\left(x_m \middle| x_n, x_k\right), \qquad k < m < n$$

$$= f(x_k, x_m, x_n) \Big/ f(x_n, x_k)$$

$$= \frac{f(x_n, x_k \mid x_m)\; f(x_m)}{f(x_k / x_n)\; f(x_n)}$$

$$= \frac{f(x_n / x_m)\; f(x_k \mid x_m)\; f(x_m)}{f(x_k / x_n)\; f(x_n)}$$

$$= \left[\frac{f(x_k \mid x_m)}{f(x_k / x_n)}\right] \frac{f(x_n, x_m)}{f(x_n)}$$

$$= \left(\frac{f(x_k \mid x_m)}{f(x_k \mid x_n)}\right) f(x_m / x_n)$$

Chapman - Kolmogorov Equation

$$f\left(x_1 \middle| x_2, x_3\right) = \frac{f(x_1, x_2, x_3)}{f(x_2, x_3)} \qquad (1)$$

$$f\left(x_2 \middle/ x_3\right) = \frac{f(x_2, x_3)}{f(x_3)} \qquad (2)$$

Multiply (1) & (2)

$$f\left(x_1 \middle| x_2, x_3\right) f\left(x_2 \middle| x_3\right) = \frac{f(x_1, x_2, x_3)}{f(x_3)}$$

$$= f\left(x_1, x_2 \middle| x_3\right)$$

Multiplying both sides by $f(x_2)$ and integrating over $-\infty \le x_2 \le \infty$

$$f\left(x_1 \middle| x_3\right) = \int_{-\infty}^{\infty} f\left(x_1, x_2 \middle| x_3\right) f(x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} f\left(x_1 \middle| x_2, x_3\right) f\left(x_2 \middle| x_3\right) dx_2 \qquad (3)$$

(Chapman - Kolmogorov rule)

$$f\left(x_n / x_k\right), \qquad n > m > k$$

$$= \int_{-\infty}^{\infty} f\left(x_n / x_m, x_k\right) f\left(x_m / x_k\right) dx_m$$

The markov property of $X(t)$ implies

$$f\left(x_n / x_m, x_k\right), \qquad n > m > k$$

$$= f\left(x_n / x_m\right)$$

Chapman - kolmogorov Equation

$$f\left(x_n / x_k\right) = \int_{-\infty}^{\infty} f\left(x_n | x_m\right) f\left(x_m | x_k\right) dx_m$$

4