Mean Square Integrability

$X(t, \omega)$, $t \in \mathbb{R}^1$, $\omega \in \Omega$ defined on $(\Omega, \mathcal{F}, P)$ is said to be MS Integrable if for $t_1 < t_2, \dots < t_n \in \mathbb{T} \subseteq \mathbb{R}$ we have :

(a) $$\lim_{n \to \infty} E\left\{\left(\sum_{i=1}^{n} X(t_i)\Delta t_i - \int_{T_1}^{T_2} X(t)\,dt\right)^2\right\} = 0$$

(b) $$\lim_{\substack{m \to \infty \\ n \to \infty}} E\left\{\left(\sum_{i=1}^{n} X(t_i)\Delta t_i - \sum_{i=1}^{m} X(t_i)\Delta t_i\right)^2\right\}$$

In otherwords

$$I_n = \sum_{i=1}^{n} X(t_i)\Delta t_i, \quad \text{where } \Delta t_i = t_i - t_{i-1}$$

converges in the MS sense to

$$\int_{T_1}^{T_2} X(t)\,dt$$

Specifically for the second interpretation we require that

$$\lim_{\substack{m \to \infty \\ n \to \infty}} E\{(I_n - I_m)^2\} = 0$$

$$E\{(I_n - I_m)^2\} = E\{I_n^2 + I_m^2 - 2I_m I_n\}$$

$$E\{I_n^2\} = E\left\{\sum_{p=1}^{n} X(t_p)\Delta t_p \sum_{q=1}^{n} X(t_q)\Delta t_q\right\}$$

$$= \sum_{p=1}^{n}\sum_{q=1}^{n} E\{X(t_p)X(t_q)\}\Delta t_p \Delta t_q$$

$$= \sum_{p=1}^{n}\sum_{q=1}^{n} R_{xx}(t_p, t_q)\Delta t_p \Delta t_q$$

$$\lim_{\substack{m\to\infty \\ n\to\infty}} E\{I_n^2\} = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{p=1}^{n}\sum_{q=1}^{n} R_{xx}(t_p, t_q)\Delta t_p \Delta t_q$$

$$= \int_{T_1}^{T_2}\int_{T_1}^{T_2} R_{xx}(t_1, t_2)\, dt_1\, dt_2$$

Similarly

$$\lim_{\substack{n\to\infty \\ m\to\infty}} E\{I_m^2\} = \sum \int_{T_1}^{T_2}\int_{T_1}^{T_2} R_{xx}(t_1, t_2)\, dt_1\, dt_2$$

&

$$\lim_{\substack{n\to\infty \\ m\to\infty}} E\{I_n I_m\} = \int_{T_1}^{T_2}\int_{T_1}^{T_2} R_{xx}(t_1, t_2)\, dt_1\, dt_2$$

Consequently,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} E\{(I_n - I_m)^2\} = 0 \quad \text{if}$$

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} R_{xx}(t_1, t_2)\, dt_1\, dt_2 < \infty$$