ECE-541, Fall 2007

Probability Theory & Stochastic Processes

Measurable Functions:

Consider now the space of functions defined on a measurable space $(\Omega, \mathcal{F})$ onto $(\mathbb{R}^1, \mathcal{B})$. Consider one such function $f(x)$ and consider further the interval $(-\infty, r)$, $\forall r \in \mathbb{R}^1$. Obviously $(-\infty, r) \in \mathcal{B}$ but in order for $f(\cdot)$ to be a measurable function its inverse image $f^{-1}(\cdot)$ needs to map $(-\infty, r)$ back into $\mathcal{F}$, i.e.,

$$\forall A \in \mathcal{B} \quad f^{-1}(A) \in \mathcal{F} \quad \text{and specifically}$$
$$\forall r \in \mathbb{R} \quad f^{-1}((-\infty, r)) \in \mathcal{F}$$

Such a map or function is measurable because a set in the transformed space can be mapped back to an event. Suppose the space under consideration is a probability space, i.e., $(\Omega, \mathcal{F}, P)$, then the transformation X will be called a random variable if $\{X^{-1}(A), A \in \mathcal{B}\} \in \mathcal{F}$. Specifically $X^{-1}((-\infty, r))$, $r \in \mathbb{R}^1 \in \mathcal{F}$

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On the notion of a random variable:

Suppose $(\Omega, \mathcal{F})$ is a measurable space and X is a transformation from $(\Omega, \mathcal{F})$ to $(\mathbb{R}^1, \mathcal{B})$ then we say that $X(\omega)$, $\omega \in \Omega$ is a random variable if $f^{-1}(A) \in \mathcal{F}$, $\forall A \in \mathcal{B}$, where $\mathcal{B}$ denotes the Borel σ-field on $\mathbb{R}^1$.

If $(-\infty, r) \in \mathcal{B}$, $\forall r \in \mathbb{R}^1$

$$\Rightarrow (-\infty, r+\tfrac{1}{k}) \in \mathcal{B}, \ \forall r \in \mathbb{R}^1, \ k \in I$$

$$\Rightarrow \bigcup_{k=1}^{\infty} (-\infty, r+\tfrac{1}{k}) \in \mathcal{B}, \ \forall r \in \mathbb{R}^1, \ k \in I$$

$$\Rightarrow (-\infty, r] \in \mathcal{B} \quad \text{(Countable Union of Borel Subsets)}$$

$$\Rightarrow (r, \infty) \in \mathcal{B}$$

$$\Rightarrow [r, \infty) \in \mathcal{B}$$

So the definition of the R.V X implies

(1) $X^{-1}((-\infty, r)) \in \mathcal{F}$, $\forall r \in \mathbb{R}^1$

(2) $X^{-1}((-\infty, r]) \in \mathcal{F}$, $\forall r \in \mathbb{R}^1$

(3) $X^{-1}([r, \infty)) \in \mathcal{F}$, $\forall r \in \mathbb{R}^1$

(4) $X^{-1}((r, \infty)) \in \mathcal{F}$, $\forall r \in \mathbb{R}^1$