Midterm Exam, Fall 2018

Probability Theory and Stochastic Processes, ECE-541
University of New Mexico
Instructor: Balu Santhanam
Date Assigned: 10/22/2018, 12:30 PM, Monday
Due Back : 10/23/2018, 12:30 PM, Tue

Instructions

1. Write clearly and legibly. Chicken scratch is hazardous to both the student and the professor.
2. Provide steps to obtain partial credit
3. It is assumed that you are aware of the UNM academic honesty policy. Needless to say copying or collaborative work of any kind will be dealt with seriously.
4. Exam is open book, open notes, you may use MATLAB to check answers.

Problem # 1.0

A continuous-time stochastic process $X(t)$ is defined by:

$$X(t, \omega) = A(\omega) + B(\omega)t + t^2,$$

where $A(\omega)$ and $B(\omega)$ are independent standard Normal random variables. For this stochastic process:

1. Plot a few sample paths or realizations.
2. Compute the mean $\mu_x(t)$ and variance $\sigma_x^2(t)$.
3. Determine the first-order PDF $f_X(x; t)$ of this process.
4. Compute the autocorrelation $R_{xx}(t_1, t_2)$, autocovariance $C_{xx}(t_1, t_2)$, and autocoherence $\rho_{xx}(t_1, t_2)$.
5. Determine the second-order joint PDF: $f_X(x_1, x_2; t_1, t_2)$. Is this process WSS? Justify your answer properly.

Problem # 2.0

A zero mean, continuous-time, Gaussian stochastic process $x(t)$ has a power spectral density of the form:

$$P_{xx}(\omega) = \frac{\omega^2 + 1}{(\omega^2 + 4)(\omega^2 + 9)}, \quad \omega \in \mathbf{R}$$

1. Compute the autocorrelation function $R_{xx}(\tau)$ of this process and its average power of this stochastic process $P_{xx}^{(\text{ave})}$.
2. Using analytic continuation, to compute the complex power spectral density $P_{xx}(s)$, its poles and zeroes, and determine the associated region of convergence.
3. Is the process ergodic in the mean? Justify your answer properly.
4. What are the first-order and second-order PDF's: $f_X(x; t)$ and $f_X(x_1, x_2; t_1, t_2)$ associated with the process $x(t)$.
5. Determine the system function $H(s)$ of the LTI system that produced the process $x(t)$ from a zero-mean white noise process. Is the solution for the system unique ?

Problem # 3.0

A M-ary *pulse amplitude modulation* (PAM) system uses a baseband transmit signal of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} a[k]p(t - kT_s),$$

where $a[k]$ are the white information symbols $a[k] \in \{\pm 1. \pm 3, \ldots \pm (M-1)\}$ and $p(t)$ is a rectangular pulse of duration 1 symbol period T_s.

1. Compute the mean and variance of the transmit signal $x_{\text{PAM}}(t)$. Is this process first-order cyclostationary? Justify your answer.

2. Compute the autocorrelation function $R_{xx}(t_1, t_2)$ of the transmit waveform. Show that the process is second-order cyclostationary.

3. Compute the cyclic autocorrelation $R_{xx}^{(n)}(\tau)$ associated with the ACF from the previous part and the corresponding cyclic PSD $P_{xx}^{(n)}(j\Omega)$. What are the associated cycle frequencies ?

4. If now the PAM signal has a random phase associated with it, i.e., the transmit waveform is:

   $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a[k]p(t - kT_b + \Theta),$$

   where $\Theta \sim U([0, T_s])$ and the symbols and the phase drift are independent. This random phase typically models timing and synchronization errors. How does the ACF and PSD result change?

Problem # 4.0

A zero-mean, WSS, continuous-time stochastic process $X(t)$ defined via:

$$X(t) = \cos(\omega_o t + \Theta(\omega)),$$

where $\Theta(\omega) \sim U([-\pi, \pi])$ is the input to a system with output:

$$Y(t) = \int_{t-T}^{t} X(\tau) d\tau.$$

For this system:

1. compute the frequency response of the underlying LTI system.
2. compute the mean $\mu_y(t)$ and variance $\sigma_y^2(t)$ of the output.
3. compute $R_{yx}(\tau)$ and $R_{yy}(\tau)$. Use Fourier transform tables as needed.
4. compute and plot the output power spectral density $P_{yy}(j\Omega)$.