Modulation of Random Processes

Let us now consider the modulation of a random process $X(t)$ with the random-phase sinusoidal signal described before :

$$Y(t) \;=\; X(t)\cos\left(\Omega_c t + \Theta\right), \tag{51}$$

where $\Theta \sim \mathbf{U}([0, 2\pi])$ and $X(t)$ and Θ are independent random variables $\forall t \;\in\; \mathbf{R}$.
The ACF of the process $Y(t)$ is given by :

$$R_{YY}(t_1, t_2) \;=\; E\left[Y(t_1)Y^*(t_2)\right] \;=\; E\left[X(t_1)\cos\left(\Omega_c t_1 + \Theta\right) X^*(t_2)\cos\left(\Omega_c t_2 + \Theta\right)\right] \tag{52}$$

Using the independence of $X(t)$ & Θ we have :

$$\begin{aligned} R_{YY}(t_1, t_2) &= E_\Theta\left[X(t_1)X^*(t_2)\right] E_\Theta\left[\cos\left(\Omega_c(t_1+t_2) + 2\Theta\right)\cos\left[\Omega_c(t_1 - t_2)\right]\right] \\ R_{YY}(t_1, t_2) &= \left(\frac{R_{XX}(t_1, t_2)}{2}\right)\left\{E_\Theta\left\{\cos\left[\Omega_c(t_1+t_2) + 2\Theta\right]\right\} + E_\Theta\left\{\cos\left[\Omega_c(t_1 - t_2)\right]\right\}\right\} \end{aligned} \tag{53}$$

Since $\Theta \sim \mathbf{U}([0, 2\pi])$ the first expectation becomes zero and then we have :

$$R_{YY}(t_1, t_2) \;=\; \frac{R_{XX}(t_1, t_2)}{2}\cos\left[\Omega_c(t_1 - t_2)\right] \tag{54}$$

This relation implies that if $X(t)$ is a WSS random process then $Y(t)$ is also a WSS process, hence :

$$R_{YY}(\tau) \;=\; \frac{R_{XX}(\tau)}{2}\cos(\Omega_c \tau) \tag{55}$$

Modulation of the $X(t)$ with the random cosine results in the modulation of the ACF of the random process. Taking the Fourier transformation on both sides of Eq. (55) :

$$P_{YY}(\Omega) \;=\; \frac{P_{XX}(\Omega + \Omega_c) + P_{XX}(\Omega - \Omega_c)}{4} \tag{56}$$

This result implies that modulation of a random process $X(t)$ with a random cosine results in a spectral upshift and a down-shift of the power spectrum $P_{XX}(\Omega)$.

Let us now look at the case where the random process $X(t)$ is modulated with a deterministic cosine. In specific let us look at the power spectrum of a *double sideband amplitude modulated* (DSB-AM) signal given by:

$$S_{DSBAM}(t) \;=\; m(t)\cos\left(\Omega_c t + \phi_o\right). \tag{57}$$

The message signal or baseband signal $m(t)$ (human-voice). Let us assume that the message signal $m(t)$ is a zero mean WSS process with ACF $R_{mm}(\tau)$. The mean of the random process $S_{DSBAM}(t)$ is therefore:

$$\mu_S(t) \;=\; E(m(t))\cos\left(\Omega_c t + \phi_o\right) \;=\; 0. \tag{58}$$

The random process $S_{DSBAM}(t)$ is therefore first-order cyclostationary. The ACF of the modulated signal denoted $R_{SS}(t, t-\tau)$ is given by:

$$R_{SS}(t, t-\tau) \;=\; \frac{R_{mm}(\tau)}{2}\left[\cos(\Omega_c \tau) + \cos\left(2\Omega_c t - \Omega_c \tau + 2\phi_o\right)\right]. \tag{59}$$

Note that this ensemble ACF $R_{SS}(t, t-\tau)$ depends on both t and τ. This process is therefore not WSS. The ensemble ACF, however, is periodic in the variable t with fundamental period $T_o = \pi/\Omega_c$. The random process $S_{DSBAM}(t)$ is therefore second-order cyclostationary with parameter T_o. We can then evaluate the time-averaged ACF denoted $\tilde{R}_{SS}(\tau)$ via:

$$\tilde{R}_{SS}(\tau) \;=\; \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} R_{SS}(t, t-\tau)dt \;=\; \frac{R_{mm}(\tau)}{2}\cos(\Omega_c\tau). \tag{60}$$

The corresponding power spectrum of DSB-AM can therefore be expressed as:

$$\tilde{P}_{SS}(\Omega) \;=\; \frac{P_{mm}(\Omega+\Omega_c) + P_{mm}(\Omega-\Omega_c)}{4}, \tag{61}$$

where $P_{mm}(\Omega)$ is the power spectrum of the baseband signal $m(t)$. As in the deterministic case, the baseband power spectrum has been up-shifted and down-shifted in frequency by Ω_c to give rise to the *upper sideband* (USB) and the *lower sideband* (LSB).

For a real message and WSS signal $m(t)$, the ACF is also real and even. It is easy to see that spectral components that are separated by $2\Omega_c = 2\pi/T_o$ are perfectly correlated (spectral redundancy). This will be of interest in co-channel communication systems where multiple users share the same carrier frequency. Note this result is analogous to the result that we obtained for the power spectrum of a random signal modulated with a random-cosine.