ECE-541
Fall 2005

More on Independence

Consider the probability space $(\Omega, \mathcal{F}, P)$ and two independent R.V's $X(\omega)$ & $Y(\omega)$ defined on this space.

$$\overset{-1}{X}\{A\} \triangleq \{\omega \ni X(\omega) \in A\} \in \mathcal{F} \qquad \text{for } \forall A, B$$
$$\overset{-1}{Y}\{B\} \triangleq \{\omega \ni Y(\omega) \in B\} \in \mathcal{F} \qquad \in \mathcal{B}$$

Independence of $X(\omega)$ & $Y(\omega)$ by definition:

$$\Pr\{\overset{-1}{X}\{A\} \cap \overset{-1}{Y}\{B\}\} = \Pr\{\overset{-1}{X}\{A\}\} \Pr\{\overset{-1}{Y}\{B\}\}$$

We are given two measurable functions g & h $\ni$

$$\left.\begin{array}{l} Z = g(X) \\ W = h(Y) \end{array}\right\} \quad g^{-1}(A) \ \& \ h^{-1}(B) \in \mathcal{B}, \ \forall A, B \in \mathcal{B}$$

We need to show that

$$\Pr\{\overset{-1}{Z}\{A\} \cap \overset{-1}{W}\{B\}\} = \Pr\{\overset{-1}{Z}\{A\}\} \Pr\{\overset{-1}{W}\{B\}\},$$
$$\forall A, B \in \mathcal{B}$$

where

$$\overset{-1}{Z}\{A\} \triangleq \{\omega \ni g(X(\omega)) \in A\} \qquad (1)$$
$$\overset{-1}{W}\{B\} \triangleq \{\omega \ni h(Y(\omega)) \in B\}$$

or

equivalently

$$\overset{-1}{Z}\{A\} \triangleq \{\omega \ni X(\omega) \in g^{-1}(A)\} \qquad (2)$$
$$\overset{-1}{W}\{B\} \triangleq \{\omega \ni Y(\omega) \in h^{-1}(B)\}$$
$$\forall A, B \in \mathcal{B}$$

$$\mathbb{P}_r\{\{\omega \ni X(\omega) \in g^{-1}(A)\}\} = \mathbb{P}_r\{\{\omega \ni g(X(\omega)) \in A\}\}$$
$$= P_r\{\{X(\omega) \in g^{-1}(A)\} = P_r\{\{g(X(\omega)) \in A\}\} \quad (3)$$

Similarly

$$\mathbb{P}_r\{\{\omega \ni X(\omega) \in h^{-1}(B)\}\} = P_r\{\{h(Y(\omega)) \in B\}\} \quad (4)$$

$$\mathbb{P}_r\{\{\omega \ni X(\omega) \in g^{-1}(A) \cap Y(\omega) \in h^{-1}(B)\}\}$$
$$= P_r\{\{g(X(\omega)) \in A \cap h(Y(\omega)) \in B\}\}$$
$$= P_r\{X(\omega) \in g^{-1}(A) \cap Y(\omega) \in h^{-1}(B)\} \quad (5)$$

By the independence of $X(\omega)$ & $Y(\omega)$

$$P_r\{X(\omega) \in g^{-1}(A) \cap Y(\omega) \in h^{-1}(B)\} = \mathbb{P}_r\{Z^{-1}(A) \cap W^{-1}(B)\}$$
$$= P_r\{X(\omega) \in g^{-1}(A)\}\, P_r\{Y(\omega) \in h^{-1}(B)\}$$
$$= \mathbb{P}_r\{\{\omega \ni g(X(\omega)) \in A\}\}\, P_r\{\{\omega \ni h(Y(\omega)) \in B\}\}$$
$$= \mathbb{P}_r\{Z^{-1}(A)\}\, \mathbb{P}_r\{W^{-1}(B)\} \quad (6)$$

$\Longrightarrow$ Events $Z^{-1}(A)$ and $W^{-1}(B)$ are independent

$\Longrightarrow$ R.V's Z & W are independent.

Note : We have used $\mathbb{P}_r$ to denote probability measure on Ω and P_r to denote induced probability measure on $\mathbb{R}$