More On Whitening

If P & Q are two covariance matrices and P is positive definite then $\exists\ V \in \mathbb{R}^{n \times n} \ni$

$$V^T P V = I \quad \text{(Whitener)}$$
$$V^T Q V = \Lambda \quad \text{(decorrelator)}$$

Proof: Since P is positive definite and a covariance matrix $\exists\ U \in \mathbb{R}^{n \times n} \ni$

$$M = U^T P U = \mathrm{diag}(\gamma_1, \gamma_2, \ldots \gamma_n),$$
where $\gamma_1, \gamma_2, \ldots \gamma_n \quad 0$ (a)

Define $Z \triangleq \mathrm{diag}(\gamma_1^{-\frac{1}{2}}, \gamma_2^{-\frac{1}{2}}, \ldots, \gamma_n^{-\frac{1}{2}})$

$$\Rightarrow \quad Z^T M Z = I \quad \text{(b)}$$
where $M = \mathrm{diag}(\gamma_1, \gamma_2 \ldots \gamma_n)$

$$\Rightarrow \quad Z^T U^T P U Z = I \quad \text{(c)}$$
or $(UZ)^T P\, UZ = I$

$\Rightarrow$ UZ whitens P

Applying UZ on Q

$$A \triangleq (UZ)^T Q (UZ) = Z^T U^T Q U Z$$

Since A is a real-symmetric matrix $\exists \; W \in \mathbb{R}^{n \times n} \ni$

$$W^T A W = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n) \quad (d)$$

Substituting for A in (d) :

$$W^T Z^T U^T Q U Z W = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n) = \Lambda$$

or $$(UZW)^T Q (UZW) = \Lambda \quad (e)$$

Since U is orthogonal & W is orthogonal

$V = UZW$ is an orthogonal matrix

$\Rightarrow$ $V = UZW$ is an orthogonal matrix that diagonalizes Q

$$(UZW)^T P (UZW)$$

$$= \underbrace{W^T Z^T U^T P U Z}_{I} W = W^T W = I$$

$\Rightarrow$ $V = UZW$ whitens P also

Since U is invertible & W is invertible and Z is invertible

$$V^{-1} = W^{-1} Z^{-1} U^{-1} = W^T Z^{-1} U^T \text{ exists}$$

Procedure for V

$$V^T P V = I \implies PV = (V^T)^{-1}$$

$$V^T Q V = \Lambda \implies QV = PV\Lambda$$

(Generalized Eigenvalue Problem)

$$Q v_i = \lambda_i P v_i$$

$$P^{-1} Q v_i = \lambda v_i$$

$\Rightarrow$ Task of finding v_i equivalent to computing eigenvectors of $P^{-1}Q$ and normalizing $v_i \ni V^T P V = I$