On Mean Squared Continuity

In class, we have see the notion of convergence in the mean-squared sense of a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. We can use this notion to formalize the concept of mean-squared continuity. A deterministic function $x(t)$ is said to be continuous both from the left and right at a point $t = t_o$ if:

$$\lim_{t \to t_o^+} x(t) = \lim_{t \to t_o^-} x(t) = x(t_o).$$

This notion can be generalized to WSS random processes by the definition that a random process $X(t)$ is *mean square continuous* at $t = t_o$ if:

$$\lim_{t \to t_o} E\{(X(t) - X(t_o))^2\} = 0.$$

From our discussion on mean squared convergence, we see that this in turn implies that the limit and the expectation operations commute, i.e.,

$$\lim_{t \to t_o} E\{X(t)\} = E\{X(t_o)\}.$$

In the general case, the condition for MS convergence can be expresses in terms of the ensemble ACF as:

$$\lim_{t \to t_o} R_{xx}(t,t) + R_{xx}(t_o,t_o) - 2R_{xx}(t,t_o) = 0$$

A sufficient condition for MS continuity of a random process $X(t)$ would be that the ACF $R_{xx}(t_1,t_2)$ be continuous along the diagonal $t_1 = t_2 = t$, i.e.,

$$\lim_{t \to t_o} R_{xx}(t,t) = \lim_{t \to t_o} R_{xx}(t,t_o) = R_{xx}(t_o,t_o).$$

If $x(t)$ is a finite average power, WSS random process we can now express this condition in terms of the autocorrelation function $R_{xx}(\tau)$ as :

$$\lim_{t \to t_o} E\{(X(t) - X(t_o))^2\} = \lim_{t \to t_o} 2R_{xx}(0) - 2R_{xx}(t - t_o).$$

This condition in turn implies that we need:

$$\lim_{\tau \to 0} R_{xx}(\tau) = R_{xx}(0).$$

This relation is significant from the perspective that to verify whether or not the WSS random process is mean-squared continuous at $t_o \in \mathbf{R}$ all we need to check for is that the autocorrelation be continuous at the origin, i.e., at $\tau = 0$. It can also be shown that continuity of the ACF, $R_{xx}(\tau)$, at $\tau = 0$ implies that $R_{xx}(\tau)$ is continuous $\forall \tau \in \mathbf{R}$.

Consider for example the Wiener process that was discussed in class. We saw that one possible model for this process was to definite it via the integral of *white Gaussian noise* (WGN):

$$W(t) = \int_0^t V(\tau) d\tau, \quad R_{vv}(\tau) = \alpha\delta(\tau), \quad V(t) \sim N(0, \alpha).$$

The ACF of the process in this case was determined to be:

$$R_{ww}(t_1,t_2) = \alpha \min(t_1,t_2) = \begin{cases} \alpha t_1 & t_1 < t_2 \\ \alpha t_2 & t_1 > t_2 \\ \alpha t & t_1 = t_2 = t \end{cases}$$

Since the sufficient condition on $R_{ww}(t_1,t_2)$ is satisfied, the Wiener process is MS continuous. On the other hand, the standard white noise process is not MS continuous because the ACF function $R_{xx}(\tau) = \sigma^2\delta(\tau)$ is not continuous at $\tau = 0$. In fact $R_{xx}(0)$ is not defined properly. The sample functions of a white noise process exhibit significant amount of jumps and are not smooth.

If for example the ACF of a WSS random process is of the form:

$$R_{xx}(\tau) = \exp\left(-a|\tau|\right), \tau \in \mathbf{R}$$

This information from the ACF allows us to deduce that the process is ergodic in the mean and is mean-squared continuous. This is quite convenient because often in practical engineering scenarios the ACF is all that we have and it conveys significant information about the underlying process. A combination of finite average power and continuity of the ACF of the process at the origin will therefore enable use to interchange the limit operation in the mean-squared sense and the expectation integral. This result will be used frequently when we look at the notion of a mean-squared derivative of a random process, mean-squared differential/difference equations and the transmission of a WSS random process through a LTI system.

As a final note, notice that a random process $X(t)$ is MS continuous then it is also continuous in probability, i.e., for any $\epsilon > 0$:

$$\lim_{t \to t_o} \Pr(|X(t) - X(t_o)| > \epsilon) = \lim_{t \to t_o} \frac{E\{(X(t) - X(t_o))^2\}}{\epsilon^2} = 0.$$