On D/A Conversion

Consider a zero-mean, WSS random sequence $x[n]$ that is input to a D/A converter. Let us specifically examine the reconstructed signal $\hat{x}_c(t)$ obtained from sinc-interpolation

$$\hat{x}_c(t) \doteq \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

Taking expectations on both sides

$$E\{\hat{x}_c(t)\} = \sum_{n=-\infty}^{\infty} \mu_x \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

$$= \mu_x \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - n\right)$$

Since the function $f(t) = 1$ is a band-limited function sampling theorem for deterministic signals holds, i.e,

$$\sum_{n=-\infty}^{\infty} 1(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right) = 1$$

Hence $E\{\hat{x}_c(t)\} = \mu_x = 0$ (In our case)

Let us now look at the variance of the reconstructed output

$$\sigma^2_{\hat{x}_c}(t) = E\left\{ \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} x[p]\, x[q] \;\mathrm{sinc}\left(\frac{t}{T_s} - p\right) \mathrm{sinc}\left(\frac{t}{T_s} - q\right) \right\}$$

$$\sigma^2_{\hat{x}_c}(t) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} r_{xx}[q-p]\, \mathrm{sinc}\left(\frac{t}{T_s} - p\right) \mathrm{sinc}\left(\frac{t}{T_s} - q\right)$$

Substitute $q - p = s$

$$\sigma^2_{\hat{x}_c}(t) = \sum_{p=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} r_{xx}[s]\, \mathrm{sinc}\left(\frac{t}{T_s} - p\right) \mathrm{sinc}\left(\frac{t}{T_s} - p - s\right)$$

$$\sigma^2_{\hat{x}_c}(t) = \sum_{s=-\infty}^{\infty} r_{xx}[s] \sum_{p=-\infty}^{\infty} \mathrm{sinc}\left(\frac{t}{T_s} - p - s\right) \mathrm{sinc}\left(\frac{t}{T_s} - p\right)$$

$$\mathrm{sinc}\left(\frac{t}{T_s}\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} T_s \Pi_W(j\Omega), \text{ where}$$

$$\Pi_W(j\Omega) = \begin{cases} 1, & |\Omega| \le \frac{\pi}{T_s} = W \\ 0, & \text{otherwise} \end{cases}$$

Since this is band-limited, sampling theorem for deterministic signals applies to the function $f_\tau(t) = \mathrm{sinc}\left(\frac{\tau - t}{T_s}\right) = f(t, \tau)$
$= f_t(\tau)$

$$f_t(\tau - sT_s) = \mathrm{sinc}\left(\frac{\tau - sT_s - t}{T_s}\right)$$

$$f_t(\tau) = \sum_{p=-\infty}^{\infty} f_t(pT_s)\, \mathrm{sinc}\left(\frac{\tau}{T_s} - p\right)$$

$$f_{pT_s}(t) = \operatorname{sinc}\left(\frac{pT_s - t}{T_s}\right)$$

$$= \operatorname{sinc}\left(\frac{t}{T_s} - p\right)$$

$$f_\tau(t) = \sum_{p=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - p\right) \operatorname{sinc}\left(\frac{\tau}{T_s} - p\right)$$

$$= \operatorname{sinc}\left(\frac{\tau - t}{T_s}\right)$$

$$\operatorname{sinc}(0) = 1 = \sum_{p=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - p\right) \operatorname{sinc}\left(\frac{t}{T_s} - p\right)$$

$$f_\tau(t - sT_s) = \sum_{p=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - s - p\right) \operatorname{sinc}\left(\frac{\tau}{T_s} - p\right)$$

$$= \operatorname{sinc}\left(\frac{\tau - t + sT_s}{T_s}\right)$$

$$\operatorname{sinc}(s) = \sum_{p=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - s - p\right) \operatorname{sinc}\left(\frac{t}{T_s} - p\right)$$

$$= \delta[s] = \frac{\sin(s\pi)}{s\pi}, \quad s \in I$$

$$\sigma^2_{\hat{x}_c}(t) = \sum_{s=-\infty}^{\infty} r_{xx}[s]\ \delta[s] = r_{xx}[0]$$

$\Longrightarrow$ $\hat{x}_c(t)$ is $n = 1^{st}$ order stationary.

Let us look at $n = 2$ order statistics

$$R_{\hat{x}_c \hat{x}_c}(t_1, t_2) = E\left\{ \hat{x}_c(t_1)\, \hat{x}_c(t_2) \right\}$$

$$= E\left\{ \sum_{p=-\infty}^{\infty} x[p] \operatorname{sinc}\left(\frac{t_1}{T_s} - p\right) \sum_{q=-\infty}^{\infty} x[q] \operatorname{sinc}\left(\frac{t}{T_s} - q\right) \right\}$$

$$R_{\hat{x}_c \hat{x}_c}(t_1, t_2) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} r_{xx}[q-p] \operatorname{sinc}\left(\frac{t_1}{T_s} - p\right) \operatorname{sinc}\left(\frac{t_2}{T_s} - q\right)$$

Substituting $q - p = \ell$

$$R_{\hat{x}_c \hat{x}_c}(t_1, t_2) = \sum_{p=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] \operatorname{sinc}\left(\frac{t_1}{T_s} - p\right) \operatorname{sinc}\left(\frac{t_2}{T_s} - p - \ell\right)$$

$$R_{\hat{x}_c \hat{x}_c}(t, t-\tau) = \sum_{p=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] \operatorname{sinc}\left(\frac{t}{T_s} - p\right) \operatorname{sinc}\left(\frac{t-\tau}{T_s} - p - \ell\right)$$

$$= \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] \sum_{p=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - p\right) \operatorname{sinc}\left(\frac{t-\tau}{T_s} - p -\right.$$

By the arguments employed for the variance

$$\sum_{p=-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T_s} - p\right) \operatorname{sinc}\left(\frac{t-\tau}{T_s} - p - \ell\right) = \operatorname{sinc}\left(\frac{\tau}{T_s} + \ell\right)$$

$$R_{\hat{x}_c \hat{x}_c}(t, t-\tau) = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] \operatorname{sinc}\left(\frac{\tau}{T_s} + \ell\right)$$

$\Longrightarrow$ $\hat{X}_c(t)$ is $n = 2$ order stationary (WSS)